

Graphs

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Outline

We discuss:

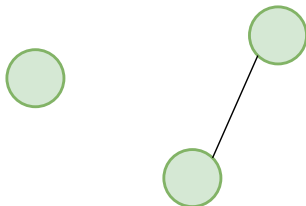
- ▶ What graphs are
- ▶ What they can be used for
- ▶ *Terminology* used with graphs

Graphs

What is a graph?

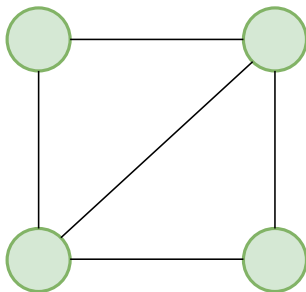
- ▶ Intuitively, we can think of a graph as being a collection of *nodes*, joined by *lines*.
- ▶ We usually call the nodes “vertices”, and the lines “edges”.
- ▶ Nodes can be joined to other nodes by one, many, or no edges.

So this is a graph:



Graph examples

And so is this:



Graph examples

And even just this:



(You can have nodes without edges; but not edges without nodes.
Edges are *defined* as joining two nodes.)

Graph definition

Formally, we define a graph by saying:

definition

A *graph* G consists of a set $V(G)$ called *vertices* together with a collection $E(G)$ of pairs of vertices.

Each pair $\{x, y\}$ in $E(G)$ is called an *edge* of G .

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- ▶ What is this set $V(G)$? Anything we like.
- ▶ It could be a set of numbers; of cities we want to visit; source files in a Java program; etc.
- ▶ Normally when we're drawing graphs, we'll label each node in the diagram to show which element of $V(G)$ it represents.

Graph example

- Suppose we have a graph G where

$$V(G) = \{A, B, C, D\}$$

and

$$E(G) = \{\{A, B\}, \{C, D\}, \{A, D\}, \{B, C\}, \{A, C\}\}$$

How many nodes and edges does G have? And what would a diagram of it look like?

Graph example

- Suppose we have a graph G where

$$V(G) = \{A, B, C, D\}$$

and

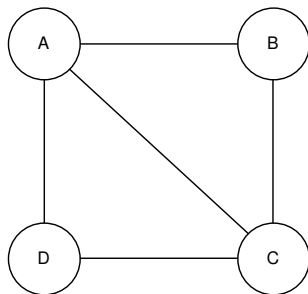
$$E(G) = \{\{A, B\}, \{C, D\}, \{A, D\}, \{B, C\}, \{A, C\}\}$$

How many nodes and edges does G have? And what would a diagram of it look like?

- G is a graph with 4 vertices and 5 edges.

Graph example

- And a diagram of it looks like this:



What are graphs used for?

- ▶ Graphs are used to model a wide range of situations
- ▶ If we know that questions about some situation correspond to particular, well-studied “standard” graph theory questions – then we have ways of solving them using graph theory.
- ▶ For instance, we might have some graphs G_1 , G_2 and G_3 defined as follows:

$V(G_1)$ = all the telephone exchanges in Australia, and $\{x, y\} \in E(G_1)$ if exchanges x and y are physically connected by fibre-optic cable.

$V(G_2)$ = all the airstrips in the world, and $\{x, y\} \in E(G_2)$ if there is a direct passenger flight from x to y .

$V(G_3)$ = all the people who have ever acted in a movie somewhere in the world, and $\{x, y\} \in E(G_3)$ if x and y have acted in a movie together.

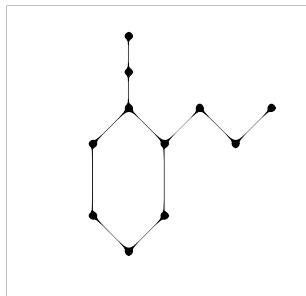
More examples

In computing:

- ▶ A graph can be used to represent processors that are connected via a communication link in a parallel computer system.

In chemistry:

- ▶ The vertices of a graph can be used to represent the carbon atoms in a molecule, and an edge between two vertices represents the bond between the corresponding atoms.



Basic properties of graphs

We will examine some of the basic terminology of graphs:

Adjacency If $\{x, y\} \in E(G)$, we say that x and y are *adjacent* to each other, and sometimes write $x \sim y$. The number of vertices adjacent to v is called the *degree* of v .

Paths A *path* of *length* n in a graph is a sequence of vertices $v_1 \sim v_2 \sim \dots \sim v_{n+1}$ such that $(v_i, v_{i+1}) \in E(G)$ and vertices $\{v_1, v_2, \dots, v_{n+1}\}$ are distinct.

Cycles A *cycle* of length n is a sequence of vertices $v_1 \sim v_2 \sim \dots \sim v_n \sim v_{n+1}$ such that $v_1 = v_{n+1}$, $(v_i, v_{i+1}) \in E(G)$ and therefore only vertices $\{v_1, v_2, \dots, v_n\}$ are distinct.

Distance The *distance* between two vertices x and y in a graph is the length of the shortest path between them.

Directed and weighted graphs

There are two important extensions to the basic definition of a graph.

Directed graphs In a directed graph, an edge is an *ordered* pair of vertices, and hence has a direction. In directed graphs, edges are often called *arcs*.

Directed Tree Each vertex has at most one directed edge leading into it, and there is one vertex (the root) which has a path to every other vertex.

Weighted graphs In a weighted graph, each of the edges is assigned a weight (usually a non-negative integer). More formally we say that a weighted graph is a graph G together with a weight function $w : E(G) \rightarrow \mathbf{R}$ (then $w(e)$ represents the weight of the edge e).

Distance in weighted graphs

When talking about weighted graphs, we need to extend the concept of distance.

definition

In a weighted graph X a path

$$x = x_0 \sim x_1 \sim \cdots \sim x_n = y$$

has *weight*

$$\sum_{i=0}^{i=n-1} w(x_i, x_{i+1}).$$

The *shortest path* between two vertices x and y is the path of minimum weight.