Generating Functions

Cheryl E Praeger

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Generating functions

Another means of organising enumeration. Two examples we have seen already.

Example 1. Binomial coefficients.

Let $X = \{1, 2, \ldots, n\}$
$c_k = \#$ k-element subsets of $X = \binom{n}{k}$.

Consider

$$F(x) = \sum_{k=0}^{n} c_k x^k.$$

This is the generating function for k-element subsets of X. We can find it explicitly:

$$F(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n.$$

Example 2. Cycle types of permutations.

Type $\underline{a} = 1^{a_1} 2^{a_2} \ldots n^{a_n}$
i.e. a_i cycles of length i, where $\sum_{i=1}^{n} a_i i = n$.

The cycle index is the generating function for cycle types. For a permutation group G

$$Z(G; x_1, \ldots, x_n) = \sum_{\underline{a}} c_{\underline{a}} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

where $c_{\underline{a}} = \#$ elements in G with cycle type $\underline{a}$.

- Generating functions are defined for any sequence $\underline{a} = (a_1, a_2, \ldots)$.
  We usually assume $\underline{a}$ is an infinite sequence.
- When $\underline{a}$ is of finite length $\underline{a} = (a_1, a_2, \ldots, a_n)$ we can define $a_i = 0$ for $i > n$.
- There can be many variables (as for the cycle index) but we will consider mainly one variable generating functions.
- Sometimes we can find the functions explicitly (as for the k-subsets) and in other cases we may only find information about their asymptotic or analytic properties.

Suppose f_n is the number of objects (of a certain kind) of "size" n. The ordinary generating function (ogf) for these objects is

$$\sum_{n\geq 0} f_n x^n$$

and the exponential generating function (egf) for these objects is

$$\sum_{n\geq 0} \frac{f_n x^n}{n!}.$$

These are called "formal power series" because we are not concerned (at least not right now) in letting x take on any particular value and we ignore (for now) questions of convergence and divergence.

Example 3. Partition function. (An ordinary generating function).

$p(n) = \#$ partitions of n

$$P(x) = 1 + \sum_{n \geq 1} p(n)x^n. \qquad \text{(Convention: } p(0) = 1.\text{)}$$

Let $p_k(n) = \#$ partitions of n with exactly k (non-zero) parts.

So $p_k(n) = 0$ for $k > n$, and for $k = 0, n \geq 1$, while $p_0(0) = 1$

$$P_k(n) = \sum_{n \geq 0} p_k(n)x^n \qquad \text{the } k\text{-part partition function.}$$

Recurrences on coefficients lead to equations on the generating functions.

Exercise. Use the recursion you proved for Assignment 2

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k)$$

to prove that

$$P_k(x) = \frac{x}{1+x^k} P_{k-1}(x).$$

You can use this to find an explicit formula for $P_k(t)$.

Examples 4. Consider the all-1 sequence $(1, 1, 1, \ldots)$.

$$\text{(ogf)} \qquad F(x) = \sum_{n=0}^{\infty} 1.x^n.$$

Note $(1 + x + x^2 + \ldots + x^n)(1 - x) = 1 - x^{n+1}$ so for $|x|$ small and $n \to \infty$

$$(1 + x + x^2 + \ldots + x^n) = \frac{1 - x^{n+1}}{1 - x} \longrightarrow \frac{1}{1 - x}$$

$$F(x) = \frac{1}{1 - x}.$$

$$\text{(egf)} \quad G(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

We can use simple generating functions like these to determine more interesting/complicated ones.

For sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$.

$$\text{ogfs: } A(x) = \sum_{n\geq 0} a_n x^n$$

$$B(x) = \sum_{n\geq 0} b_n x^n$$

Addition:

$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n) x^n, \text{ the ogf for } (a_n + b_n)_{n\geq 0}.$$

Multiplication:

$$A(x).B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n,$$

the ogf for $\left(\sum_{k=0}^{n} a_k b_{n-k}\right)_{n\geq 0}$.

egfs: $A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n, \qquad B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$

Addition:

$$A(x) + B(x) = \sum_{n \geq 0} \frac{a_n + b_n}{n!} x^n, \text{ the egf for } (a_n + b_n)_{n \geq 0}.$$

Multiplication:

$$A(x).B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n$$

$$= \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) x^n$$

the egf for $\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right)_{n \geq 0}$.

We can also do calculus: differentiate, take logs etc.

Example 5. Derangements: $d(n) = \#$ derangements on $\{1, 2, \ldots, n\}$.

Permutations with k fixed points are derangements on the remaining $n-k$ points.
For a given k there are $\binom{n}{k}$ subsets of size k and hence $\binom{n}{k}d(n-k)$ permutations with exactly k fixed points. Hence

$$n! = \sum_{k=0}^{n} \binom{n}{k} d(n-k).$$

Form the egf for the sequence $(n!)_{n\geq 0}$.

$$F(x) = \sum_{n\geq 0} \frac{n!}{n!} x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

But we also have

$$F(x) = \sum_{n\geq 0} \frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k} d(n-k)\right) x^n$$

$$= A(x).D(x)$$

where $A(x)$ is the egf for $(1)_{n\geq 0}$, i.e. $A(x) = e^x$ and $D(x)$ is the egf for $\big(d(n)\big)_{n\geq 0}$.

Hence $D(x) = \dfrac{F(x)}{A(x)} = \dfrac{1}{1-x}\cdot\dfrac{1}{e^x} = \dfrac{e^{-x}}{1-x}$.

We can use this to find $d(n)$ by expanding both sides and equating coefficients.

$$\sum_{n\geq 0} \frac{d(n)}{n!} x^n = \frac{1}{1-x}.e^{-x} = \left(\sum_{n\geq 0} x^n\right)\left(\sum_{n\geq 0} \frac{(-x)^n}{n!}\right)$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) x^n.$$

So, equating coefficients:

$$d(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This alternative derivation is sometimes simpler than combinatorial or "Pólya-theoretic" methods.

The Partition Function

Infinite products occur and are useful — the partition function

$$P(t) = \sum_{n\geq 0} p(n)t^n$$

where $p(0) = 1$ (convention) and $p(n) = \#$ partitions of n.

Each partition of n consists of a_i parts of size i $(i = 1, \ldots, n)$ where each $a_i \geq 0$ and $a_1.1 + a_2.2 + \ldots + a_n.n = n$.

So: $p(n) =$ the coefficient of t^n in

$$\prod_{i\geq 1}(1 + t^i + t^{2i} + t^{3i} + \ldots).$$

The Partition Function continued

To simplify this product recall

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

So taking $x = t^i$, $1 + t^i + t^{2i} + \dots = \frac{1}{1-t^i}$.

Hence $p(n) =$ coeff. of t^n in $\prod_{i\geq 1} \frac{1}{1-t^i}$.

So

$$P(t) = \prod_{i\geq 1} \frac{1}{1-t^i}$$

This is a famous result going back to Euler.

The Partition Function continued

The i^{th} factor is $1 + t^i + t^{2i} + \dots$

Note: coefficient of t^{ij} is 1 and this is the number of partitions of ij with all parts of size i.

The partition function $P(t)$ is the product over all i of the (rather trivial) generating functions $P_i(t) = 1 + t^i + t^{2i} + \dots$ for the partitions with all parts of size i.

We can use this observation in several ways.

First way: To get at $p_{\mathrm{odd}}(n) = \#$ partitions of n with all odd parts. We just take the $P_i(t)$ with i odd so

$$P_{\mathrm{odd}}(t) = \sum_{n \geq 0} p_{\mathrm{odd}}(n) t^n = \prod_{i \text{ odd}} \frac{1}{1 - t^i}$$

Second way: Restricting multiplicities.
Suppose we only want partitions in which all parts have distinct sizes; i.e. a_i parts of size i where each $a_i = 0$ or 1 and $\sum a_i i = n$.

Let $p_{\text{unequal}}(n) = \#$ partitions of n with distinct parts.
Then $p_{\text{unequal}}(n)$ is the coefficient of t^n in $\prod_{i\geq 1} \frac{1}{1+t^i}$.
So

$$\begin{aligned} P_{\text{unequal}}(t) &= \sum_{n\geq 0} p_{\text{unequal}}(n) t^n \\ &= \prod_{i\geq 1} (1+t^i). \end{aligned}$$

You can experiment with many other restrictions e.g. all parts even, no part occurs more than k times, etc.

A Bizarre Coincidence

In the expression for $P_{\text{unequal}}(t)$ write each $1+t^i$ as $\frac{1-t^{2i}}{1-t^i}$ so

$$P_{\text{unequal}}(t) = \prod_{i \geq 1} \frac{1-t^{2i}}{1-t^i}.$$

Every $1-t^{2i}$ in the numerator (i^{th} term) cancels with $1-t^{2i}$ in the denominator ($2i^{th}$ term) and leaves

$$P_{\text{unequal}}(t) = \prod_{i \text{ odd}} \frac{1}{1-t^i} = P_{\text{odd}}(t).$$

This slightly hand-waving proof is hard to believe, so here is a "bijective" "combinatorial proof".

Take a partition λ of n into odd parts

$$\lambda : a_{2i-1} \text{ parts of size } 2i-1.$$

Define a new partition $\mu = \mu(\lambda)$ depending on and determined by λ as follows.

Write each positive integer (uniquely) as $2^k j$ where $k \geq 0$ and j is odd.

μ has either zero or 1 part of size $2^k j$ (for each $k \geq 0$ and odd j), namely μ has a part of size $2^k j \Leftrightarrow a_j$ contains the term 2^k in its binary expansion.

Examples

$n = 7$ $\lambda : 1 + 3 + 3$
$a_1 = 1,\ a_3 = 2$

So μ has 1 part of size $2^0.1 = 1$
and 1 part of size $2^1.3 = 6$

$n = 19$ $\lambda : 1 + 3 + 3 + 3 + 3 + 3 + 3$
$a_1 = 1,\ a_3 = 6 = 2 + 4$

Here μ has 1 part of size $2^0.1 = 1$
and 1 part of size $2^1.3 = 6$
and 1 part of size $2^2.3 = 12$

Always $\mu = \mu(\lambda)$ is a partition of n with distinct parts, and the correspondence $\lambda \longmapsto \mu(\lambda)$ is $1-1$ and onto.

Why is μ a partition of n?

$$n = \sum_{i \text{ odd}} a_i i \text{ by definition of } \lambda.$$

Let $a_i = 2^{k_{i_1}} + 2^{k_{i_2}} + \dots$ (binary expansion).

Then $n = \sum_{i,j} 2^{k_{i_j}}.i =$ sum of sizes of parts of μ.

Why is $\lambda \longmapsto \mu(\lambda)$ $1-1$?

Suppose $\mu(\lambda) = \mu(\lambda')$ and λ' has a_i' parts of size i, $\forall$ odd i.

By definition of μ : μ has a part of size $2^k j \Leftrightarrow 2^k$ occurs in binary expansion of a_j (and a_j').

This means a_j and a_j' have exactly the same binary expansion, so $a_j = a_j'$ for all j. So $\lambda = \lambda'$.

Why is $\lambda \longmapsto \mu(\lambda)$ onto?

Take any partition of n into distinct parts μ.

For each part $2^k j$ (j odd)
"put 2^k into a_j."

i.e. for each odd j, define $a_j =$ sum of all 2^k such that μ has a part of size $2^k j$.

There are also generating function *proofs* for results where no natural combinatorial proofs are known.

- the number of self-complementary digraphs on $2n$ vertices is equal to the number of self-complementary graphs on $4n$ vertices.
- the number of self-complementary graphs on n vertices is equal to the difference between the numbers of graphs with an even number of edges and an odd number of edges on n vertices.

Unlabelled graphs and their connected components

$g_n = \#$ unlabelled graphs on n vertices
$c_n = \#$ connected graphs on n vertices.

Consider ogf for these

$$C(t) = \sum_{n\geq 1} c_n t^n, \quad G(t) = \sum_{n\geq 0} g_n t^n \quad (g_0 = 1).$$

Each unlabelled graph on n vertices arises as a union of its connected components, say c_Γ copies of each connected unlabelled graph Γ where each $c_\Gamma \geq 0$ and $\sum_\Gamma c_\Gamma n(\Gamma) = n$.
Here $n(\Gamma) = \#$ vertices of Γ.

So $g_n =$ the coefficient of t^n in

$$\prod_{\Gamma}(1 + t^{n(\Gamma)} + t^{2n(\Gamma)} + \dots)$$

where the product is over all finite unlabelled graphs Γ. This product is equal to

$$\prod_{n\geq 1}(1 + t^{n} + t^{2n} + \dots)^{c_n}$$

since for each n there are c_n factors $(1 + t^n + \dots)$, one for each of the c_n unlabelled connected graphs on n vertices.

This function is equal to $\displaystyle\prod_{n\geq 1}\frac{1}{(1-t^n)^{c_n}}$. Thus

$$G(t) = \prod_{n\geq 1}\frac{1}{(1-t^n)^{c_n}}.$$

Remark: We can find g_n (Pólya theory) and solve for the c_n, one by one.

Question: How might we find $C(t) = \sum_{n \geq 1} c_n t^n$ from this? Or at least see an equation relating $C(t)$ and $G(t)$.

$$\log G(t) = -\sum_{n \geq 1} c_n \log(1 - t^n)$$

by an extension of the properties of "$\log$". (This can be proved.)

Then using the Taylor expansion of $\log$:

$$\begin{aligned}
\log G(t) &= -\sum_{n\geq 1} c_n \log(1-t^n) \\
&= \sum_{n\geq 1} c_n (\sum_{j\geq 1} \frac{t^{jn}}{j}) \\
&= \sum_{j\geq 1} \frac{1}{j} (\sum_{n\geq 1} c_n (t^j)^n) \\
&= \sum_{j\geq 1} \frac{C(t^j)}{j}
\end{aligned}$$

Hence

$$G(t) = \exp(\sum_{j\geq 1} \frac{C(t^j)}{j}).$$

Exercises I

Exercise 1: Use the recursion you proved for Assignment 2

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k)$$

to prove that

$$P_k(x) = \frac{x}{1+x^k} P_{k-1}(x).$$

Hence find an explicit formula for $P_k(t)$.

Exercise 2: The Fibonacci Numbers are defined by $F_0 = 1$, $F_1 = 1$, $F_2 = 2$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

(a) Find an equation satisfied by the ogf for $(F_n)_{n\geq 1}$.

(b) Solve it to find the ogf.

(c) (Link with partitions.) Show that F_n is the number of *ordered* partitions of n with all parts equal to 1 or 2. [e.g. the ordered partitions $1+2$ and $2+1$ of 3 are regarded as different.]

Exercises II

Exercise 3: (Exercise 11 on p70 of Cameron's book)

(a) Let $s(n)$ be the number of sequences $(x_1, \ldots, x_k)$ of integers satisfying $1 \le x_i \le n$ for all i and $x_{i+1} \ge 2x_i$ for $i = 1, \ldots, k-1$. (The length of the sequence is not specified; in particular, the empty sequence is included.) Prove the recurrence

$$s(n) = s(n-1) + s(\lfloor \frac{n}{2} \rfloor)$$

for $n \ge 1$, with $s(0) = 1$. Calculate a few values of s. Show that the generating function $S(t)$ satisfies $(1-t)S(t) = (1+t)S(t^2)$.

Exercises III

Exercise 3 continued:

(b) Let $u(n)$ be the number of sequences $(x_1, \ldots, x_k)$ of integers satisfying $1 \leq x_i \leq n$ for all i and $x_{i+1} > \sum_{j=1}^{i} x_j$ for $i = 1, \ldots, k-1$. Calculate a few values of u. Can you discover a relationship between s and u? Can you prove it?

Exercises IV

Exercise 4: Let $g_n(m) = \#$ labelled graphs on n vertices with m edges.
Define the ogf for $g_n(m)$ by

$$G_n(x) = \sum_{m \geq 0} g_n(m) x^m.$$

Recursion: Let $\mathcal{S}_{n,m}$ be the set of labelled graphs on n vertices with m edges.
Define a map ϕ from X to Y, where

$$\begin{aligned} X &= \{(e, \Gamma) \,|\, \Gamma \in \mathcal{S}_{n,m+1}, e \in E(\Gamma)\} \\ Y &= \{(e', \Gamma') \,|\, \Gamma' \in \mathcal{S}_{n,m}, e \notin E(\Gamma')\}, \end{aligned}$$

by $\phi : (e, \Gamma) \longmapsto (e, \Gamma \backslash e)$ where $\Gamma \backslash e$ denotes the graph Γ with the edge e removed.

Exercises V

Exercise 4 continued:

For example, if Γ is the triangle on vertex set $\{1,2,3\}$, then $\phi : (\{1,2\},\Gamma) \to (\{1,2\},P)$ where P is the path with edges $\{1,3\}$, and $\{2,3\}$.

(a) Prove that ϕ is a well-defined bijection.

(b) Hence prove that

$$(m+1)\, g_n(m+1) = \left(\binom{n}{2} - m \right) g_n(m).$$

Exercises VI

Exercise 4 continued:

(c) Multiply by x^m and add over all $m \geq 0$ to obtain

$$\sum_{m\geq 1}(m+1)\, g_n(m+1)x^m = \binom{n}{2}G_n(x) - \sum_{m\geq 0} m g_n(x)x^m.$$

What is this? Differentiate $G_n(x)$ with respect to x:

$$G_n'(x) = \sum_{m\geq 1} m g_n(m)x^{m-1}.$$

(d) Prove that $G_n'(x) = \binom{n}{2}\frac{G_n(x)}{1+x}$ and deduce that $G_n(x) = (1+x)^{\binom{n}{2}}$. Can you also give a more direct proof of this result?