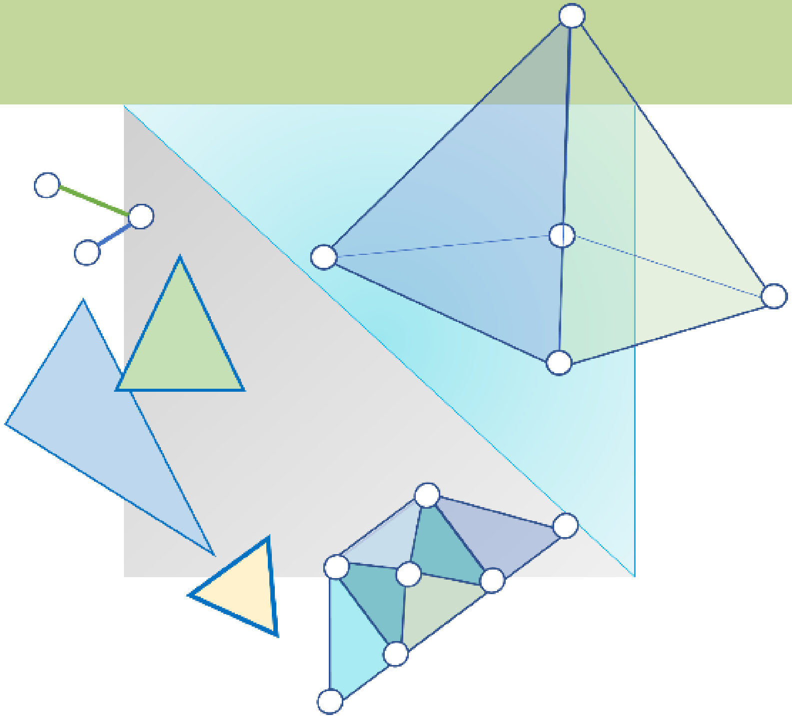
CITS3003 Graphics & Animation

Lecture 7:
Representation and Coordinate Systems

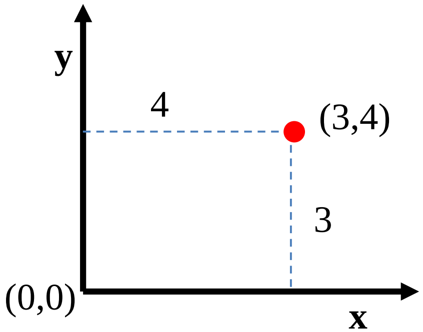


Content

- Intro. to Geometric objects
- Intro. to scalar field, vectors spaces and affine spaces
- Dimensionality and linear independence
- Discuss change of frames and bases
- Intro. to coordinate frames

Geometric Objects

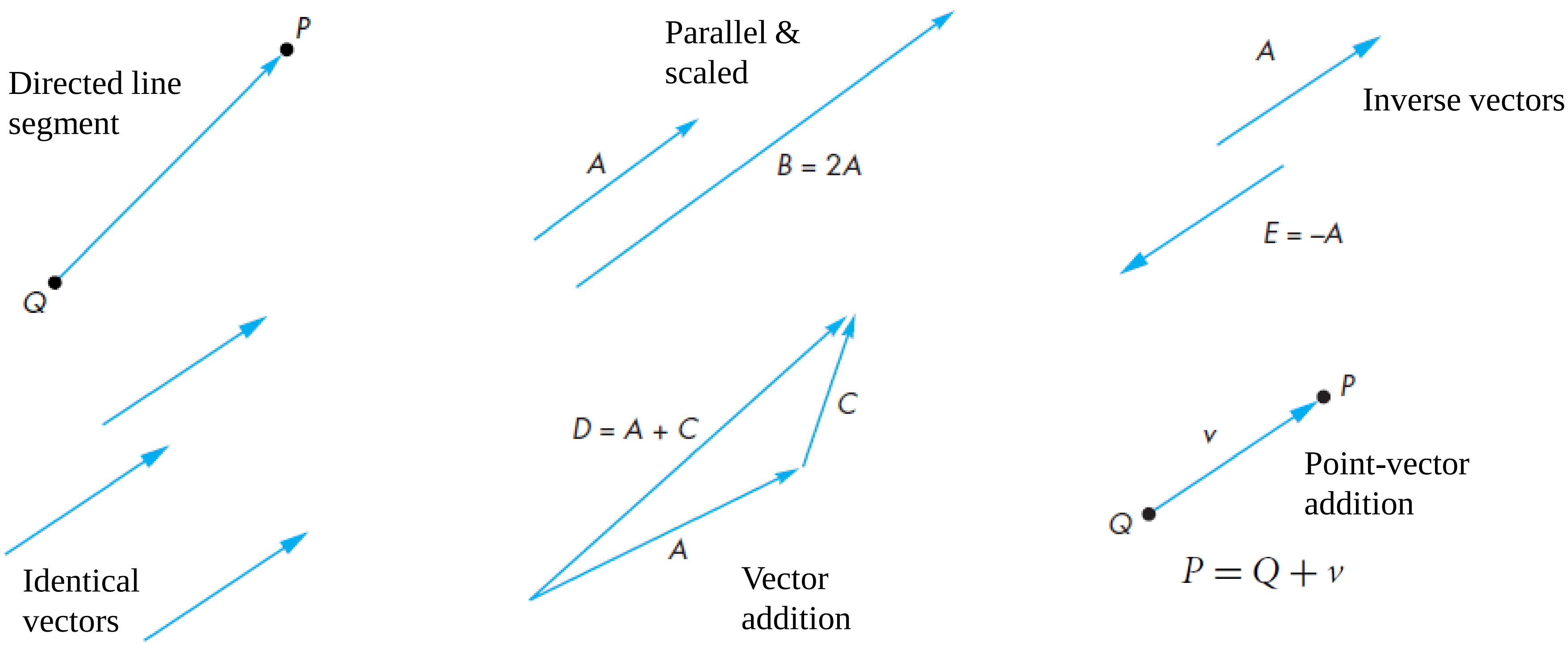
- **Point** (fundamental geometric object)
  - Location in space/coordinate system
    - Example: Point (3, 4)
  - Cannot add or scale points
  - mathematical point has neither a size nor a shape



- **Scalars**
  - Real /complex numbers
  - Used to specify quantities
  - Obey a set of rules
    - addition and multiplication
    - commutivity and associativity *// $a + b = b + a$; $(a + b) + c = a + (b + c)$*
    - multiplicative and additive inverses *// $a + (-a) = 0$; $a \cdot a^{-1} = 1$*

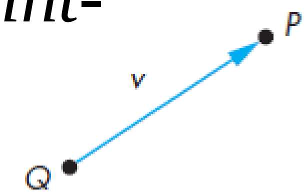
Geometric Objects

- Vector
  - Is any quantity with direction and magnitude
    - e.g., Force, velocity etc.
  - Can be added, scaled and rotated
  - A vector does not have a fixed location in space



Vector-Point Relationship

- Vector
  - Looking at things differently, two points can be thought of defining a vector, i.e., *point-point-subtraction* $\boldsymbol{v} = \boldsymbol{P} - \boldsymbol{Q}$
    - *Subtract 2 Points = vector*
    - *Point + vector =point*



  - Because vectors can be multiplied by scalars, expressions, below make sense

$$\boldsymbol{P} + 3\boldsymbol{v}$$

Point-vector addition

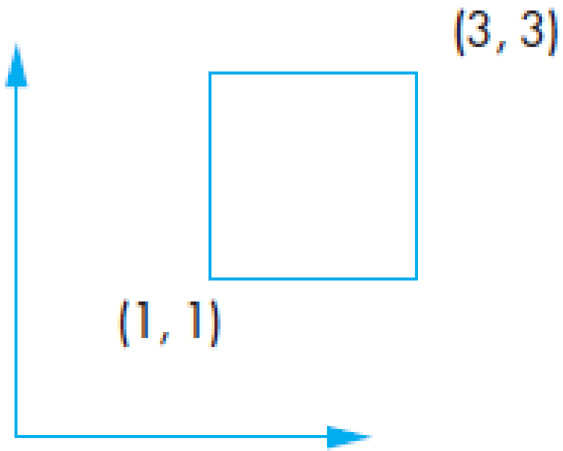
$$2\boldsymbol{P} - \boldsymbol{Q} + 3\boldsymbol{v}$$

$\boldsymbol{P} + (\boldsymbol{P} - \boldsymbol{Q}) + 3\boldsymbol{v}$

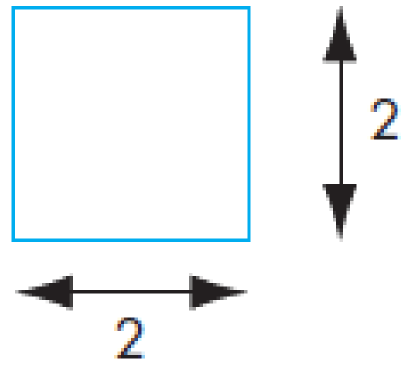
  - But this does not $\boldsymbol{P} + 3\boldsymbol{Q} - \boldsymbol{v}$

Coordinate Free Geometry

Points exist in space regardless of any reference or coordinate system



Object in a Coordinate System



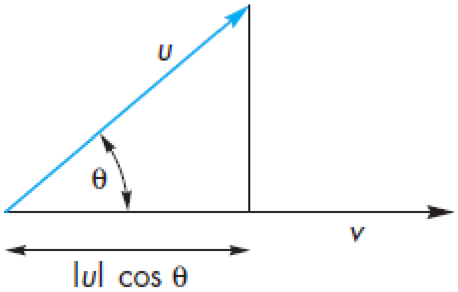
Object without a Coordinate System

Spaces

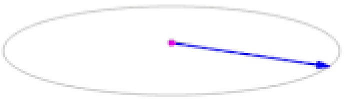
- **Scalar field**
  - A pair of scalars can be combined to form another scalar
    - two operations: *addition* and *multiplication*
  - obey the closure, associativity, commutivity, and inverse properties
- **Vector space**
  - Contains vectors and scalars
  - Vector-scalar and vector-vector interactions
  - *Euclidean vector space*
    - is an extension of a vector space that adds a measure of size or distance
    - e.g., length of a line segment
  - *Affine vector space*
    - Extension of vector space and includes “point”
    - Vector-point addition and point-point subtraction are possible
    - No point-point addition and point-scalar operation are possible

Dot and Cross Products

- Dot (inner) product
  - Square of magnitude $|u|^2 = u \cdot u.$
  - If $u.v = 0$, u and v are orthogonal
  - dot product $\cos\theta = \frac{u \cdot v}{|u||v|}$
  - Orthogonal projection $|u| \cos\theta = u \cdot v / |v|$



- Cross (outer) product
  - Given by $u \times v = |u||v| \sin(\theta)$
  - Normal $n = u \times v.$



Linear Independence

- A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is *linearly independent* when

  $$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + .. \ a_n\mathbf{v}_n = \mathbf{0} \text{ iff } a_1 = a_2 = \ldots = 0$$

- If a set of vectors is *linearly independent*, we cannot represent one vector in terms of the other vectors
- If a set of vectors is *linearly dependent*, at least one can be written in terms of the others

Examples

- Independent:
  - $v1=[1,2]^T$, $v2=[-5,3]^T$
- Dependent:
  - $v1=[2,-1,1]^T$, $v2=[3,-4,2]^T$, $v3=[5,-5,3]^T$

Linear Independence (cont.)

- For example: Let

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

  then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of linearly independent vectors.

- What are the values of $\alpha_1, \alpha_2,$ and α_3 if we want $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$?

Linear Independence (cont.)

- What are the values of $\alpha_1, \alpha_2,$ and α_3 if we want $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$?
- That is, we want

$$\alpha_1\begin{pmatrix}5\\0\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\3\\0\end{pmatrix} + \alpha_3\begin{pmatrix}0\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Dimension

- In a vector space, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space
- In an *n*-dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, any vector $\mathbf{w}$ can be written as

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$$

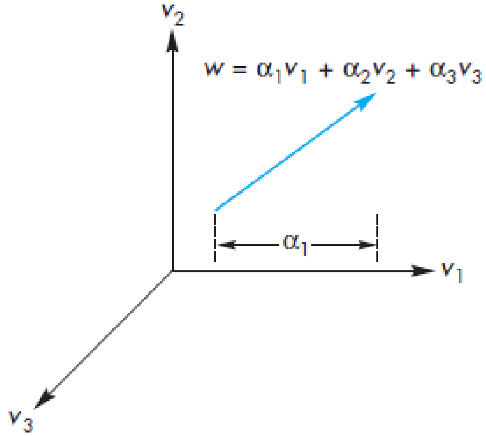
where the coefficients $\{a_i\}$ are unique and are called representations of **w**

Dimension (cont.)

- Let us define a basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
- The vector

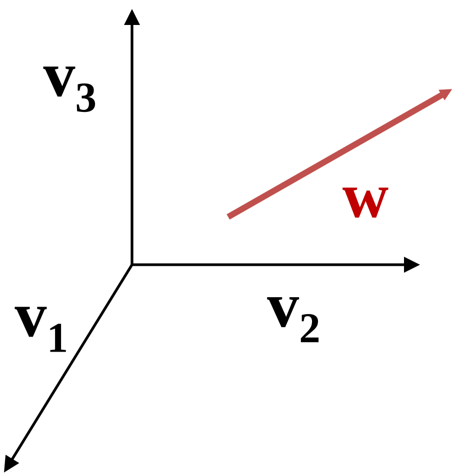
$$\mathbf{w} = \begin{pmatrix} 10.5 \\ 21.3 \\ 0.9 \end{pmatrix}$$

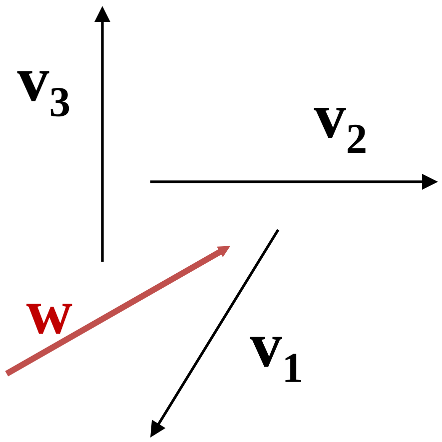
can be written as $\mathbf{w} = 10.5\,\mathbf{v}_1 + 21.3\,\mathbf{v}_2 + 0.9\,\mathbf{v}_3$ and the coefficients $\alpha_1 = 10.5, \alpha_2 = 21.3,$ and $\alpha_3 = 0.9$ are unique



Coordinate Systems (cont.)

- Which one is correct?





- Both are correct, because **vectors have no fixed location**

Coordinate Systems

- Until now we have been able to work with geometric entities without using any frame of reference, such as a coordinate system
- We need a frame of reference to relate points and objects to our physical world.
  - For example, where is a point? We can't answer this without a reference system
  - The same point can be represented in its
    - World coordinates
    - Camera coordinates

Coordinate Systems

- Consider a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$
- A vector $\mathbf{w}$ is written $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$
- The list of scalars $\{a_1, a_2, \ldots\ a_n\}$ is the *representation* of $\mathbf{w}$ with respect to the given basis
- We can write the representation as a row or column array of scalars

$$\boldsymbol{\alpha} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^{\mathrm{T}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Coordinate systems (cont.)

- For example, let $\mathbf{v} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$. If $\mathbf{v}_1 = [1 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{v}_2 = [0 \quad 1 \quad 0]^{\mathrm{T}}$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^{\mathrm{T}}$, then $\boldsymbol{\alpha} = [2 \quad 3 \quad -4]^{\mathrm{T}}$
- Note that this representation is <u>with respect to a particular basis</u>

Change of Coordinate System

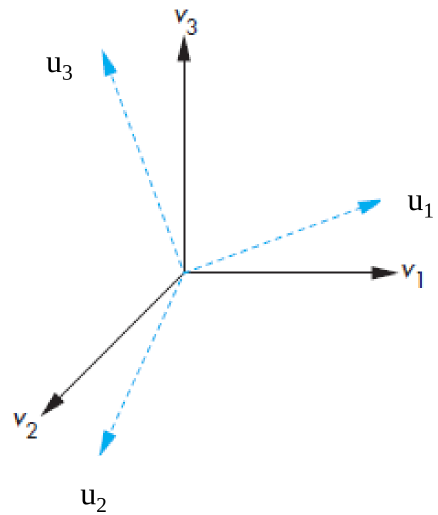
- Let's consider transformation of two bases
  - {v1, v2, v3} and {u1, u2, u3} are two bases.
  - Each basis vector in the second set can be represented in terms of the first basis

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3,$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3,$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3.$$

The 3 × 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$\Rightarrow$ $$\mathbf{u} = \mathbf{M}\mathbf{v}.$$



Change of Coordinate Systems

- Consider the <u>same</u> vector $\mathbf{w}$ with respect to two different coordinate systems having basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Suppose that

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$$

$$\mathbf{w} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3$$

- Then the representations are:

$$\mathbf{a} = [\alpha_1 \quad \alpha_2 \quad \alpha_3]^{\mathrm{T}}$$

$$\mathbf{b} = [\beta_1 \quad \beta_2 \quad \beta_3]^{\mathrm{T}}$$

- Equivalently,

$$\mathbf{w} = \mathbf{a}^{\mathrm{T}}\mathbf{v}$$

$$\text{and } \mathbf{w} = \mathbf{b}^{\mathrm{T}}\mathbf{u}$$

Change of Coordinate System

- Thus,

$$\mathbf{w} = \mathbf{b}^{\mathrm{T}}\mathbf{u} = \mathbf{b}^{\mathrm{T}}\mathbf{M}\,\mathbf{v} = \mathbf{a}^{\mathrm{T}}\,\mathbf{v}$$

- and

$$\boldsymbol{a} = \boldsymbol{M}^{T}\boldsymbol{b}$$

- Also

$$\boldsymbol{b} = \boldsymbol{T}\boldsymbol{a} \quad \text{where,} \quad \boldsymbol{T} = (\boldsymbol{M}^{T})^{-1}$$

Change of Coordinate System

- Example,

Suppose **u** and **v** are two basis related to each other as follows:

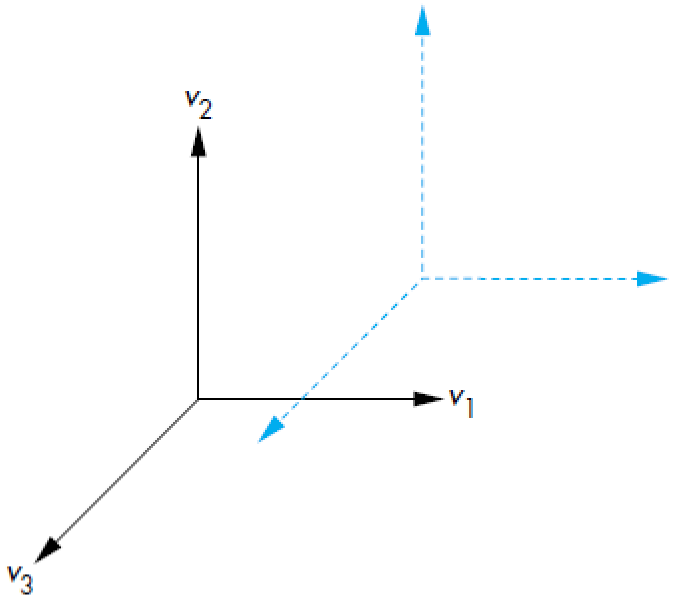
$$u_1 = v_1,$$
$$u_2 = v_1 + v_2,$$
$$u_3 = v_1 + v_2 + v_3.$$

We have a representation vector **a** (below) that is represented in **v**, what will be its representation in **u**

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Coordinate Frame

- We can also do all this in coordinate systems:



Further Reading

“Interactive Computer Graphics – A Top-Down Approach with Shader-Based OpenGL” by Edward Angel and Dave Shreiner, 6th Ed, 2012

- Sec 3.3 *Coordinate Systems and Frames* (all subsections)
- Sec 3.4 *Frames in OpenGL*