1. Show that for two neuron Hopfield network with asynchronous net-value update the corresponding energy function (with no bias)

\[ E(x_1, x_2) = \frac{1}{2} \sum_{i,j=1}^{2} \omega_{ij} x_i x_j \]  

is not necessarily non-increasing if one of the diagonals \( w_{ii} \) is negative. Hint: consider the case when \( w_{12} = w_{21} = 3/4, w_{11} = -1 \) and \( w_{22} = 0 \), which still renders the symmetric matrix. Calculate the net value after asynchronously updating the first neuron value and compare \( E(x_1, x_2) \) and \( E(x'_1, x'_2) \). Check it for the training vector \((1, 1)\). Assume that each \( x_i \) takes one of the binary values \( \{0, 1\} \).

2. Show that for the synchronous dynamics used in Hopfield net the symmetry of the synaptic matrix \( W \) is a critical condition for the existence of the cycles with at most length 2. Hint: consider the two neuron Hopfield net with \( w_{12} = 1, w_{21} = -1, w_{11} = w_{22} = 0 \) and take the training vector \((1, -1)\). Apply the synchronous update and guess the cycle length. Assume that the binary values are from \( \{-1, 1\} \). Draw the corresponding states of the net.

3. Show that for the asynchronous dynamics used in Hopfield net the symmetry of the synaptic matrix \( W \) is a critical condition for the non-existence of the cycles. Hint: consider the two neuron Hopfield net with \( w_{12} = 1, w_{21} = -1, w_{11} = w_{22} = 0 \) and take the training vector \( x = (1, -1) \). Apply the asynchronous update and guess the existence of the cycle length. Show also that for the asynchronous dynamics the negative diagonal may also produce cycles. Hint: assume now \( w'_0 = -1, w'_{22} = 1, w_{12} = w_{21} = 0 \) and the training vector coincides with \( x \). Assume for both cases binary values from \( \{-1, 1\} \). Draw the corresponding states of the net.

4. Consider the three neuron Hopfield net with the synaptic matrix \( W \) defined as: \( w_{11} = w_{22} = w_{33} = 0, w_{21} = w_{12} = w_{23} = w_{32} = -2/3, w_{31} = w_{13} = 2/3 \). Show that two training vectors \( x_1 = (1, -1, 1) \) and \( x_2 = (-1, 1, -1) \) are the only stable points of the corresponding net. Apply the asynchronous update scheme. Assume that the binary values are from \( \{-1, 1\} \). Show on a diagram what happens with the remaining 6 vectors. What is their Hamming distance from the corresponding stable state. Recover the synaptic matrix \( W \) from both stable vectors \( x_1 \) and \( x_2 \).

5. Consider the three neuron Hopfield net with the synaptic matrix \( W \): \( w_{11} = w_{22} = w_{33} = 0, w_{21} = w_{12} = w_{23} = w_{32} = 1, w_{31} = w_{13} = -1 \). Find the next state \( x(1) \) for the vector \( x(0) = (1, 1, 1) \) by using synchronous update scheme. Compare the energies \( E(x(0)) \) and \( E(x(1)) \) by using formula (1) (for \( n = 3 \)). Why \( E(x(0)) < E(x(1)) \) does not contradict the general theory. Assume that net vales are from from \( \{0, 1\} \). Fix the energy to non-increasing.

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