GENG2140
Modelling and Computer Analysis for Engineers

Lectures 37 - 38:
Numerical solution of ODEs – boundary value problems
Content

• Case study – a loaded beam
• The finite differences method
• Matlab solutions to the beam case study
Case study – a loaded beam:

Physical models of beams:

Beams are structural members used to transmit bending moment and transverse shear.

Free end – can translate and rotate

Simply supported end - can rotate but can not translate

Clamped end – can not rotate or translate

We have boundary conditions at both ends of the beam.
load(z) – intensity of lateral load (per unit length) [N/m]

\[
\begin{align*}
\frac{d}{dz} \text{shear}(z) &= \text{load}(z); \quad \text{shear}(z) - \text{shear force} \\
\frac{d}{dz} M(z) &= \text{shear}(z); \quad M(z) - \text{bending moment} \\
\frac{d}{dz} \text{slope}(z) &= \frac{M(z)}{E I(z)}; \quad E I(z) - \text{flexural rigidity} \\
\frac{d}{dz} u(z) &= \text{slope}(z); \quad u(z) - \text{deflection}
\end{align*}
\]
• By combining the first 2 equations:

\[ \frac{d^2}{dz^2} M(z) = \text{load}(z) \]

• From last 2 equations:

\[ M(z) = EI(z) \frac{d^2u(z)}{dz^2} \]

• => the equation describing the deflection of the beam:

\[ \frac{d^2}{dz^2} \left( EI(z) \frac{d^2u(z)}{dz^2} \right) = \text{load}(z) \]

= a 4\textsuperscript{th} order differential equation

We make a change of variables to transform it into a system of 1\textsuperscript{st} order ODEs
The system of ODEs becomes:

\[
\begin{align*}
  u(z) &= y(1) \\
  \text{slope}(z) &= y(2) \\
  M(z) &= y(3) \\
  \text{shear}(z) &= y(4)
\end{align*}
\]

- The system of ODEs becomes:

\[
\begin{align*}
  y_1' &= y_2 \\
  y_2' &= \frac{y_3}{EI(z)} \\
  y_3' &= y_4 \\
  y_4' &= \text{load}(z)
\end{align*}
\]
The finite differences method:

- The fundament of the finite difference method is to replace the derivatives with finite differences.
- A number of mesh points are distributed along the beam.

\[ y'_i = \frac{y_{i-1} - y_{i+1}}{2\Delta z}, \text{ or better} \]

\[ y'_{i+\frac{1}{2}} = \frac{y_{i+1} - y_i}{\Delta z}, \text{ with } y_{i+\frac{1}{2}} = \frac{y_i + y_{i+1}}{2} \]
• By replacing in the system of first order ODEs:

\[
\begin{align*}
\frac{1}{\Delta z} (y_{1,i+1} - y_{1,i}) &= \frac{1}{2} (y_{2,i+1} + y_{2,i}) \\
\frac{1}{\Delta z} (y_{2,i+1} - y_{2,i}) &= \frac{1}{2} \left( y_{3,i+1} \frac{1}{Ei_{i+1}} + y_{3,i} \frac{1}{Ei_{i}} \right) \\
\frac{1}{\Delta z} (y_{3,i+1} - y_{3,i}) &= \frac{1}{2} (y_{4,i+1} + y_{4,i}) \\
\frac{1}{\Delta z} (y_{4,i+1} - y_{4,i}) &= \frac{1}{2} (load_{i+1} + load_{i})
\end{align*}
\]

• If we have M+1 points in the mesh and write the above equations for i=1...M => 4*M equations with 4*M+4 unknowns => we need 4 boundary conditions to obtain a unique solution
Example – cantilever beam:

\[
\begin{align*}
\frac{1}{\Delta z} y_{1,i+1} - \frac{1}{\Delta z} y_{1,i} - \frac{1}{2} y_{2,i} - \frac{1}{2} y_{2,i+1} &= 0 \\
\frac{1}{\Delta z} y_{2,i+1} - \frac{1}{\Delta z} y_{2,i} - \frac{1}{2} \frac{1}{E I_i} y_{3,i} - \frac{1}{2} \frac{1}{E I_{i+1}} y_{3,i+1} &= 0 \\
\frac{1}{\Delta z} y_{3,i+1} - \frac{1}{\Delta z} y_{3,i} - \frac{1}{2} y_{4,i} - \frac{1}{2} y_{4,i+1} &= 0 \\
\frac{1}{\Delta z} y_{4,i+1} - \frac{1}{\Delta z} y_{4,i} &= \frac{1}{2} \left( \text{load}_{i+1} + \text{load}_i \right) 
\end{align*}
\]

\[
\begin{align*}
u(z) &= y(1) \\
slope(z) &= y(2) \\
M(z) &= y(3) \\
shear(z) &= y(4)
\end{align*}
\]

\[\begin{align*}
y_{1,1} &= 0 \\
y_{2,1} &= 0 \\
y_{3,M+1} &= 0 \\
y_{4,M+1} &= 0
\end{align*}\]

\[i=1 \ldots M\]

\Rightarrow 4M+4 equations and 4M+4 unknown
\[ A \cdot y = L, \text{ with} \]

\[
y = \begin{bmatrix} y_{1,1} \\ y_{2,1} \\ y_{3,1} \\ y_{4,1} \\ y_{1,2} \\ \vdots \\ y_{4,M+1} \end{bmatrix}
\]
• In reality the number of mesh points can be very large (> 10^6) – the storage of coefficients becomes a problem, as well as the computation time
• Special techniques are used in such cases – sparse matrices storage and computations
• Because each equation contains up to 4 unknowns, each row in the coefficient matrix A has only up to 4 non-zero coefficients => the matrix of coefficients is almost totally filled with zeros (is very sparse)
• Only the non-zero coefficients are stored and used in computations
• Example coefficient matrix for the cantilever beam, M=50:
  – 204x204 elements
  – 704 elements stored
• If the original ODE to be solved is non-linear => the resulting system of algebraic equations is nonlinear and is usually solved in several iterations

• For example, if the proportional limit for the young modulus is exceeded => $E=E(u) \leftrightarrow E=E(y_1)$

• In general, the non-linear system of equations can be expressed as:

$$A(y) \cdot y = L(y)$$

— This is a large non-linear system of equations
• Possible solution method for large non-linear systems of equations:

\[ A(y) \cdot y = L(y) \]

1. Consider \( A \) constant and solve as a linear system
   \[ \Rightarrow y \]
2. Substitute \( y \) into \( A(y) \) and get a new estimate of the coefficient matrix
3. Repeat from step 1 until \( \| y_{\text{new}} - y_{\text{previous}} \| < \varepsilon \)
Thank you!