Partitions and Permutations

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Partitions

The word *partition* is shared by (at least) two different concepts, although both refer to the process of dividing an object into smaller sub-objects.

- **Integer Partitions**
  A partition of an integer $n$ is a way to write it as a sum of smaller integers, such as
  
  $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$.

- **Set Partitions**
  A partition of a *set* is a way to divide it into a number of subsets, such as
  
  $\{1, 2, 3\}, \{1, 2\}\{3\}, \{1, 3\}\{2\}, \{1\}\{2, 3\}, \{1\}\{2\}\{3\}$
Lexicographic order

We will represent a partition of an integer $n$ by a sequence $a_1 a_2 \ldots$, where $a_1 \geq a_2 \geq \cdots > 0$. Thus the partitions of 7 would be represented as follows.

\begin{align*}
1111111 & \quad 211111 \quad 2211 \\
2221 & \quad 3111 \quad 3211 \\
322 & \quad 331 \quad 4111 \\
421 & \quad 43 \quad 511 \\
52 & \quad 61 \quad 7
\end{align*}

In this table the partitions are given in lexicographic order.
Reverse Lexicographic Order

The simplest algorithm to generate partitions actually generates them in reverse lexicographic order, starting with the partition \( n \) and ending with

- Find the largest index \( i \) such that \( a_i \neq 1 \); suppose that \( a_i = a + 1 \) so that
  \[
  \alpha = \beta(a + 1)11\ldots1.
  \]

- Replace the suffix \((a + 1)11\ldots1\) by \( aa\ldots ar\) where \( r < a \), thus obtaining
  \[
  \text{succ}(\alpha) = \beta aa\ldots ar.
  \]
Example 1

What is the successor of $\alpha = 22211$?

- The largest $i$ such that $a_i \neq 1$ is $i = 3$

\[
\begin{array}{cccccc}
a_1 & a_2 & a_3 & a_4 & a_5 \\
2 & 2 & 2 & 1 & 1 \\
\end{array}
\]

- Replace the suffix 211 with 1111

\[
\begin{array}{ccccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
2 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Therefore we get

\[\text{succ}(22211) = 221111.\]
Example 2

What is the successor of \( \alpha = 3311 \)?

1. The largest \( i \) such that \( a_i \neq 1 \) is \( i = 2 \)

\[
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
3 & 3 & 1 & 1 \\
\end{array}
\]

2. Replace the suffix 311 with as many 2s as possible, and a remainder if necessary.

\[
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
3 & 3 & 1 & 1 \\
3 & 2 & 2 & 1 \\
\end{array}
\]

Therefore we get

\[
succ(3311) = 3221.
\]
Fixed number of parts

Suppose we are interested in generating the number of partitions of $n$ into exactly $k$ parts, for example if $n = 11$ and $k = 4$ we get

$$
\begin{align*}
3332 & \quad 4322 & \quad 4331 & \quad 4421 \\
5222 & \quad 5321 & \quad 5411 & \quad 6221 \\
6311 & \quad 7211 & \quad 8111 & 
\end{align*}
$$
**Successors**

The easiest algorithm for generating partitions of fixed size $k$ starts with the partition

$$(n - k + 1)1111$$

and then computes the successor of $\alpha = a_1a_2 \ldots a_k$ as follows:

- Let $i$ be the smallest index such that $a_i < a_1 - 1$.
- Assign $a_i + 1$ to each of $a_2, a_3, \ldots, a_i$ and then set $a_1$ as needed to maintain a partition of $n$.

The algorithm terminates if there are no values $i$ such that $a_i < a_1 - 1$; in this case each part of the partition is $\left\lfloor \frac{n}{k} \right\rfloor$ or $\left\lceil \frac{n}{k} \right\rceil$. 
Example

Suppose that $n = 15$ and $k = 5$ and we want the successor of $\alpha = 66111$.

▶ The smallest index $i$ such that $a_i < 5$ is $i = 3$

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 \\
  6 & 6 & 1 & 1 & 1 \\
\end{array}
\]

▶ Set $a_2$ and $a_3$ to the value $a_3 + 1$ which is 2

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 \\
  * & 2 & 2 & 1 & 1 \\
\end{array}
\]

▶ Set $a_1$ to the required value to maintain a partition of $n = 15$

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 \\
  9 & 2 & 2 & 1 & 1 \\
\end{array}
\]
Coplex ordering

This ordering of partitions with a fixed number of parts is not reverse lexicographic, but rather colexicographic.

<table>
<thead>
<tr>
<th>Lexicographic</th>
<th>Reverse lex</th>
<th>Colexicographic</th>
</tr>
</thead>
<tbody>
<tr>
<td>3322</td>
<td>7111</td>
<td>7111</td>
</tr>
<tr>
<td>3331</td>
<td>6211</td>
<td>6211</td>
</tr>
<tr>
<td>4222</td>
<td>5311</td>
<td>5311</td>
</tr>
<tr>
<td>4321</td>
<td>5221</td>
<td>4411</td>
</tr>
<tr>
<td>4411</td>
<td>4411</td>
<td>5221</td>
</tr>
<tr>
<td>5221</td>
<td>4321</td>
<td>4321</td>
</tr>
<tr>
<td>5311</td>
<td>4222</td>
<td>3331</td>
</tr>
<tr>
<td>6211</td>
<td>3331</td>
<td>4222</td>
</tr>
<tr>
<td>7111</td>
<td>3322</td>
<td>3322</td>
</tr>
</tbody>
</table>
Conjugate permutations

Suppose that $f$ and $g$ are two permutations in the symmetric group $\text{Sym}(n)$. Then they are said to be *conjugate* if

$$f = h^{-1}gh$$

for some $h \in \text{Sym}(n)$.

**Example**

The permutations $f = (1, 2, 3)(4, 5)$ and $g = (1, 3, 4)(2, 5)$ are conjugate because

$$(1, 2, 3)(4, 5) = (2, 3, 4) (1, 3, 4)(2, 5) (2, 4, 3)$$

and therefore we may take $h = (2, 4, 3)$ in the above definition.
Conjugacy is an equivalence relation

- Every permutation is conjugate to itself
  \[ f = e^{-1}fe \]

- If \( f \) is conjugate to \( g \), then \( g \) is conjugate to \( h \)
  \[ f = h^{-1}gh \implies g = hfh^{-1} \]

- If \( a \) is conjugate to \( b \) and \( b \) conjugate to \( c \), then \( a \) is conjugate to \( c \)
  \[ a = h^{-1}bh \text{ and } b = g^{-1}cg \implies a = h^{-1}g^{-1}cgh = (gh)^{-1}c(gh). \]
Conjugacy Classes

This means that the permutations fall into *disjoint* classes called *conjugacy classes* such that two permutations are conjugate if and only if they lie in the same class.

<table>
<thead>
<tr>
<th></th>
<th>(3, 4)</th>
<th>(1, 2)(3, 4)</th>
<th>(2, 3, 4)</th>
<th>(1, 2, 3, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>(2, 3)</td>
<td>(1, 3)(2, 4)</td>
<td>(2, 4, 3)</td>
<td>(1, 2, 4, 3)</td>
</tr>
<tr>
<td></td>
<td>(2, 4)</td>
<td>(1, 4)(2, 3)</td>
<td>(1, 2, 3)</td>
<td>(1, 3, 4, 2)</td>
</tr>
<tr>
<td></td>
<td>(1, 2)</td>
<td></td>
<td>(1, 2, 4)</td>
<td>(1, 3, 2, 4)</td>
</tr>
<tr>
<td></td>
<td>(1, 3)</td>
<td></td>
<td>(1, 3, 2)</td>
<td>(1, 4, 3, 2)</td>
</tr>
<tr>
<td></td>
<td>(1, 4)</td>
<td></td>
<td>(1, 3, 4)</td>
<td>(1, 4, 2, 3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1, 4, 2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(1, 4, 3)</td>
<td></td>
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</tbody>
</table>
Cycle Structure

The cycle structure of a permutation is the number of cycles of each length in its cycle decomposition.

Theorem
Two permutations are conjugate in $\text{Sym}(n)$ if and only if they have the same cycle structure.

We need to prove two things

- If two permutations are conjugate, then they have the same cycle structure, and
- If two permutations have the same cycle structure, then they are conjugate.
Conjugate permutations have the same cycle structure

If the permutation $g$ maps

\[ i \rightarrow ig \]

then $h^{-1}gh$ maps

\[ ih \rightarrow igh. \]

Therefore for every cycle

\[ (a, b, \ldots, c) \]

in the cycle decomposition of $g$, there is a corresponding cycle

\[ (ah, bh, \ldots, ch) \]

in the cycle decomposition of $f = h^{-1}gh$. 
Permutations with the same cycle structure are conjugate

If two permutations $f$ and $g$ have the same cycle structure, then we can find a conjugating permutation $h$ such that $f = h^{-1}gh$.

For each cycle

$$(f_1, f_2, \ldots f_k)$$

of $f$ there is a corresponding cycle

$$(g_1, g_2, \ldots, g_k)$$

of $g$.

Then define $h$ by the rule

$$h : g_i \rightarrow f_i.$$
Example

Let

\[ f = (1, 4, 5)(2, 7)(3, 6) \quad g = (2, 4, 6)(1, 5)(3, 7) \]

Then define \( h \) (in image notation) as follows:

\[
h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 7 & 5 & 6 \end{pmatrix}
\]

Then \( h = (1, 2)(5, 7, 6) \) and it is easy to check that

\[ f = h^{-1}gh \]

and so \( h \) is a conjugating permutation.
In GAP

gap> f := (1,4,5)(2,7)(3,6);
(1,4,5)(2,7)(3,6)
gap> g := (2,4,6)(1,5)(3,7);
(1,5)(2,4,6)(3,7)
gap> h := PermList([2,1,3,4,7,5,6]);
(1,2)(5,7,6)
gap> h^-1*g*h;
(1,4,5)(2,7)(3,6)
gap> f = last;
true
gap>
Cycle structure

The cycle structure of a permutation can be represented in a number of different ways. One way would be to simply list the lengths of the cycles. For example, if \( f \in \text{Sym}(10) \) has cycle decomposition

\[
f = (1, 2, 4)(5, 6)(8, 9, 10)
\]

then its cycle structure could be written as

\[
33211
\]

In this representation, the cycle structure of a permutation is simply a partition of \( n \), the degree of the permutation!
Partitions and Conjugacy Classes

We have therefore essentially proved the following:

**Theorem**
The number of conjugacy classes of $\text{Sym}(n)$ is equal to the number of partitions of $n$.

For $\text{Sym}(4)$ the correspondence is

<table>
<thead>
<tr>
<th>Partition</th>
<th>Representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111</td>
<td>$e$</td>
</tr>
<tr>
<td>211</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>22</td>
<td>$(1, 2)(3, 4)$</td>
</tr>
<tr>
<td>31</td>
<td>$(1, 2, 3)$</td>
</tr>
<tr>
<td>4</td>
<td>$(1, 2, 3, 4)$</td>
</tr>
</tbody>
</table>
Counting partitions

Let \( p(n) \) be the number of partitions of \( n \), and let \( p_k(n) \) be the number of partitions of \( n \) that have \( k \) parts. Then

\[
p(n) = \sum_{k=1}^{n} p_k(n).
\]

We can write these numbers out in a triangle reminiscent of Pascal’s triangle:

\[
\begin{array}{cccc}
p_1(1) & \ & \ & \\
p_1(2) & p_2(2) & \ & \\
p_1(3) & p_2(3) & p_3(3) & \\
p_1(4) & p_2(4) & p_3(4) & p_4(4) \\
\cdots & \cdots & \cdots & \cdots
\end{array}
\]
Top of the triangle

By direct calculation we see that the triangle starts

\[
\begin{array}{cccccc}
 & & & 1 & & \\
 & 1 & & 1 & & \\
1 & 1 & & 1 & & \\
1 & 2 & & 1 & & \\
1 & 2 & 2 & & 1 & \\
1 & 3 & 3 & 2 & & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
\end{array}
\]

The total number of partitions $p(n)$ of each number $n$ is obtained by summing the entries in each row of this triangle.
A recurrence

We can express the values of $p_k(n)$ in terms of smaller such numbers by splitting up the partitions of $n$ according to the number of parts that are equal to 1. So if $P$ is the set of all partitions of $n$ with $k$ parts, then define $P_0, P_1, \ldots, P_k$ by

$$P_i = \{ \pi \in P \mid \pi \text{ has exactly } i \text{ 1s} \}$$

Then clearly

$$|P| = \sum_{i=0}^{k} |P_i|.$$ 

Therefore we need to count the number of partitions in each set $P_i$. 
Partitions with $i$ 1s

Given a partition $\pi \in P_i$, consider the partition obtained by subtracting one from each part of $\pi$. This is a partition of $n - k$ with $k - i$ parts, and conversely any partition of $n - k$ with $k - i$ parts yields a partition in $P_i$ if we add 1 to each part, and adjoin $i$ parts equal to 1.

Therefore

$$|P_i| = p_{k-i}(n - k).$$

Example

The partitions of 10 into 4 parts with two 1s are

$$4411 \hspace{1cm} 5311 \hspace{1cm} 6211$$

and the partitions of 6 into 2 parts are

$$33 \hspace{1cm} 42 \hspace{1cm} 51.$$
Therefore

\[ p_k(n) = \sum_{i=0}^{k} p_{k-i}(n - k) \]

and replacing \( i \) by \( k - i \) we get

\[ p_k(n) = \sum_{i=0}^{k} p_i(n - k). \]

This sum involves terms of the form \( p_0(x) \); this value must be taken to be 0 unless \( x = 0 \) in which case \( p_0(0) \) is defined to be 1.
Extending the triangle

We will use this to extend the triangle to $n = 7$

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 \\
1 & 3 & 3 & 2 & 1 & 1 \\
1 & 3 & 4 & 3 & 2 & 1 & 1 \\
\end{array}
\]

The blue value $p_3(7)$ is obtained by adding the first 3 values on the line for $n = 7 - 3 = 4$, thus

$$4 = 1 + 2 + 1.$$
Size of conjugacy classes

Given a partition of $n$, how many permutations have that cycle structure?

For example, how big is the conjugacy class of $\text{Sym}(7)$ with cycle structure 322? A permutation with this cycle structure has the form

$$(a, b, c)(d, e)(f, g).$$

There are $7!$ ways of assigning the numbers 1, 2, \ldots, 7 to the 7 positions, but many of them yield the same permutation.

For example

$$(1, 2, 3)(4, 5)(6, 7) \text{ and } (2, 3, 1)(6, 7)(4, 5)$$

are the same.
Overcounting

For the conjugacy class of $\text{Sym}(7)$ with cycle structure 322, there are

- 3 ways to write the cycle $(a, b, c)$
- 2 ways to write the cycle $(d, e)$
- 2 ways to write the cycle $(f, g)$
- 2 ways to order the two 2-cycles

Therefore every permutation arises in $3 \times 2 \times 2 \times 2 = 24$ ways, and so the total number of distinct permutations in this conjugacy class is $7!/24 = 210$. 
In general

In general, suppose that a partition has $c_i$ parts equal to $i$, so that

$$c_1 + 2c_2 + 3c_3 + \cdots + nc_n = n.$$  

Then the number of permutations in the conjugacy class with this cycle structure is

$$\frac{n!}{(\prod_i c_i!i^{c_i})}.$$  

Proof.

Each cycle of length $i$ can be written in $i$ different ways, and the $c_i$ cycles of length $i$ can be written in $c_i!$ different orders. Hence the denominator of this expression counts the number of different ways of writing the permutation.
Check that this all works for $n = 5$ where there are 7 partitions:

<table>
<thead>
<tr>
<th>Partition</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>Overcount</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>11111</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>120</td>
<td>1</td>
</tr>
<tr>
<td>2111</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>221</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>311</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>32</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>41</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>120</td>
</tr>
</tbody>
</table>
Stirling Numbers

The *Stirling numbers* are the two series of numbers \( s(n, k) \) and \( S(n, k) \) defined as follows:

- **Stirling numbers of the 1st kind:**
  \[ (-1)^{n-k} s(n, k) \] is the number of permutations of degree \( n \) with \( k \) cycles.

- **Stirling numbers of the 2nd kind:**
  \( S(n, k) \) is the number of partitions of an \( n \)-set into \( k \) non-empty parts.
First, the second kind

It is easy to list all the set partitions for small $n$ and $k$:

- $n = 1$
  - $k = 1$: 1

- $n = 2$
  - $k = 1$: 12
  - $k = 2$: 1|2

- $n = 3$
  - $k = 1$: 123
  - $k = 2$: 12|3, 13|2, 1|23
  - $k = 3$: 1|2|3

- $n = 4$
  - $k = 1$: 1234
  - $k = 2$: 1|234, 134|2, 124|3, 123|4, 12|34, 13|24, 14|23
  - $k = 3$: 12|3|4, 13|2|4, 14|2|3, 1|23|4, 1|24|3, 1|2|34
  - $k = 4$: 1|2|3|4
The triangle

We can write the numbers out in a triangle (similar to Pascal’s triangle) as follows:

\[
\begin{array}{cccc}
S(1,1) \\
S(2,1) & S(2,2) \\
S(3,1) & S(3,2) & S(3,3) \\
S(4,1) & S(4,2) & S(4,3) & S(4,4) \\
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

From the previous slide we get

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
A recurrence

We can express the Stirling number $S(n, k)$ in terms of smaller Stirling numbers by noting that a set partition of $\{1, \ldots, n\}$ with $k$ parts is obtained either by

- Adding $n$ as a singleton to a partition of $\{1, \ldots, n - 1\}$ with $k - 1$ parts, or
- Putting $n$ into one of the cells of a partition of $\{1, \ldots, n - 1\}$ with $k$ parts

Counting the number of set partitions of each type yields

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).$$
Illustration

For \((n, k) = (4, 3)\), this formula is

\[
S(4, 3) = S(3, 2) + 3 \times S(3, 3)
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 3 & 3 \times 1 \\
1 & 7 & 6 & 1 \\
\vdots & \vdots & \vdots & \vdots
\end{array}
\]

and the corresponding set partitions in the two groups are

- \(12|3|4, 13|2|4, 1|23|4\)
- \(14|2|3, 1|24|3, 1|2|34\)
An explicit formula

We can find an explicit formula for \( S(n, k) \) by using the principle of inclusion and exclusion:

Let \( A \) denote the set of all functions

\[
f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, k\}
\]

Now, let \( A_i \) be the set

\[
A_i = \{ f \in A \mid f^{-1}(i) = \emptyset \}.
\]

In other words, \( A_i \) is the set of functions whose range does not include \( i \).
Principle of Inclusion/Exclusion

If $A_1, A_2, \ldots, A_k$ are all subsets of a set $A$ then for any index set $\mathcal{I} \subseteq \{1, 2, \ldots, k\}$, define

$$A_{\mathcal{I}} = \bigcap_{i \in \mathcal{I}} A_i$$

(where we take $A_{\emptyset} = X$).

Principle of Inclusion/Exclusion

The number of elements of $X$ that do not belong to any of the sets $A_1, A_2, \ldots, A_k$ is given by

$$\sum_{\mathcal{I} \subseteq \{1,2,\ldots,k\}} (-1)^{|\mathcal{I}|} |A_{\mathcal{I}}|.$$
Set partitions with \( k \) parts

By PIE, the number of functions that map \( \{1, 2, \ldots, n\} \) onto \( \{1, 2, \ldots, k\} \) is given by:

\[
\sum_{\mathcal{I} \subseteq \{1,2,\ldots,k\}} (-1)^{|\mathcal{I}|} |A_{\mathcal{I}}|.
\]

For each possible size \( 0 \leq j \leq k \) there are \( \binom{k}{j} \) possible subsets \( \mathcal{I} \) of size \( j \), and each of them makes the same contribution to this sum. More precisely, if \( |\mathcal{I}| = j \) then

\[
|A_{\mathcal{I}}| = (k - j)^n
\]

because this just counts the number of functions that avoid a particular set of \( j \) elements.
The formula

Putting this together, we see that the number of functions from \{1, \ldots, n\} onto \{1, \ldots, k\} is given by

\[
\sum_{j=0}^{j=k} (-1)^j \binom{k}{j} (k - j)^n
\]

which (on replacing \(j\) by \(k - j\)) is equal to

\[
\sum_{j=1}^{j=k} (-1)^{k-j} \binom{k}{j} j^n.
\]

Now each such function determines a partition of \{1, 2, \ldots, n\} into \(k\) parts, but we have counted each partition \(k!\) times and so

\[
S(n, k) = \frac{1}{k!} \sum_{j=1}^{j=k} (-1)^{k-j} \binom{k}{j} j^n.
\]
Bell numbers

The total number of partitions of a set of size $n$ into any number of non-empty parts is called the Bell number $B(n)$. Thus

$$B(n) = \sum_{k=1}^{k=n} S(n, k).$$

The first few Bell numbers are

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975$$

GAP has built-in functions Stirling1(n,k), Stirling2(n,k) and Bell(n) giving these numbers.
Now, the first kind

The triangle of Stirling numbers of the first kind

\[
s(1, 1) \\
s(2, 1) \quad s(2, 2) \\
s(3, 1) \quad s(3, 2) \quad s(3, 3) \\
s(4, 1) \quad s(4, 2) \quad s(4, 3) \quad s(4, 4) \\
\cdots \quad \cdots \quad \cdots \quad \cdots
\]

starts as follows:

\[
\begin{array}{cccc}
1 & & & \\
-1 & 1 & & \\
2 & -3 & 1 & \\
-6 & 11 & -6 & 1 \\
\cdots & \cdots & \cdots & \cdots
\end{array}
\]
A recurrence

We can find a recurrence expressing \( s(n, k) \) in terms of smaller Stirling numbers, by noting that a permutation of degree \( n \) with \( k \) cycles can be obtained either by

- Adjoining element \( n \) as a fixed point to a permutation of degree \( n - 1 \) with \( k - 1 \) cycles, or
- Inserting \( n \) into one of the cycles of a permutation of degree \( n - 1 \) with \( k \) cycles.

and then counting the number of permutations in each category.
Finding $s(4, 2)$

The permutations of degree 4 with 2 cycles consist of

- The permutations of degree 3 with 1 cycle, with 4 adjoined as a fixed point
  
  $$ (1, 2, 3)(4) \quad (1, 3, 2)(4) $$

- The permutations of degree 3 with 2 cycles, with 4 inserted into one of the cycles
  
  $$(1, 2)(3) \quad (1, 3)(2) \quad (1)(2, 3)$$
  $$ (1, 4, 2)(3) \quad (1, 4, 3)(2) \quad (1, 4)(2, 3) $$
  $$ (1, 2, 4)(3) \quad (1, 3, 4)(2) \quad (1)(2, 4, 3) $$
  $$ (1, 2)(3, 4) \quad (1, 3)(2, 4) \quad (1)(2, 3, 4) $$

There are 2 in the first group and $9 = 3 \times 3$ in the second group, thus giving us 11 altogether.
The formula

This arguments shows us that

$$|s(n, k)| = |s(n - 1, k - 1)| + (n - 1)|s(n - 1, k)|$$

but it does not take into account the sign of $s(n, k)$ (that is, whether it is positive or negative).

Now $(-1)^{n-k}$ is equal to $(-1)^{(n-1)-(k-1)}$ and so taking the signs into consideration we get

$$s(n, k) = s(n - 1, k - 1) - (n - 1)s(n - 1, k).$$
An astonishing connection

The reason for associating signs with the Stirling numbers is related to the following astonishing connection between these two sets of numbers, illustrated here for $n = 4$.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
2 & -3 & 1 & 0 \\
-6 & 11 & -6 & 1
\end{pmatrix} = I_4
\]