Mathematical notation and writing

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Mathematical Writing

- Writing is difficult.
- Writing Mathematics is very difficult.
- Keep your prose simple and direct. Use plain English.
- Learn through feedback.

Reference: Handbook of Writing for the Mathematical Sciences, Nicholas J. Higham.

Notation is important

- There are rules for mixing a symbolic language with English; these rules are well-accepted by the scientific community.
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- Your notation influences the readability of your document.
- Too little notation can be cumbersome.
- Too much notation can be confusing.

Notation is important

"It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain."

Notation is important

OR

$$x^n + y^n = z^n$$

where x, y, z and n are integers, has no non-zero solutions for $n > 2$.

Available Symbols:

Roman, Greek, old German, Hebrew and other alphabets, and several punctuation marks.
Some letters and symbols are standardised:

$$\begin{array}{ccc} e & \infty & \pi \\ \prod & \equiv & \delta_{ij} \\ i & n! & |z| \\ \sum & \binom{n}{k} & O(f(n)) \end{array}$$

Avoid using these symbols in a different way!

Implicit Meanings

Some symbols have implicit meanings:
i, j, k, m, n often denote integers or i, j denote $\sqrt{-1}$.
ε, δ usually represent a small real number.

Bad Notation

For any $n \in \mathbb{R}$ we can find an $\varepsilon \in \mathbb{Z}$ such that $\varepsilon > n$.

Correct mathematically but increases the cognitive load on the reader.

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Read papers to know what notation is standard in your field.

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Use the same type of symbol (roman/greek, lower/upper case) for the same type of object.

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Let $g \in S$. [...] Now take $G \in S$.

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One symbol one meaning

Avoid using one symbol for different things, especially in close proximity,
e.g. if $\mathbb{P}$ denotes a set of primes do not use $\mathbb{P}$ to denote a probability law later on.

Do not start a sentence with a symbol

Absolute rule:

Do not start a sentence with a symbol. The editor will make you change it every time.

Can be confusing.

Example

The data structure A is very powerful and can represent many types of images. A can represent an image of a Coke can. A can is a difficult image object to interpret because of the specularity of the metal.

Write complete sentences

Use words!!!

- In preference to symbols such as $\Rightarrow$, $\exists$, $\forall$ (unless you are writing specifically about formal logic). Symbols do not abbreviate sentences!

Bad examples

- For all elements $\in \mathbb{Z}$ ...
- Then any set of integers $\neq \mathbb{Z}$
- a number ≤ 3

Write complete sentences

Use words!!!

- Do not simply list formulas one after another. Each formula is read as a phrase in English, so they should be tied together with words.

Write complete sentences

Use words!!!

- Even between two formulas:

Example

Since $p \geq 0, p \in U$. Hard to read.

Since $p \geq 0$, it follows that $p \in U$. Better.

Example

Let P be a set of n points M_i, $i = 1, \ldots, n$ of ∂X.

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Let P be a set of n points M_i, $i = 1, \ldots, n$ of ∂X.
Let P be a set of n points M_i, where $i = 1, \ldots, n$ and each M_i is a member of ∂X.

More examples

Example

... for any $1 \leq i \leq 10$,
Instead write
... for any i with $1 \leq i \leq 10$.

Example

Therefore this holds for all integers ≥ 3.
Instead write
Therefore this hold for all integers greater than 2.

Unnecessary superscripts or subscripts

Superscripts and subscripts can be very useful, but always ask yourself: can I do without them.

Example

Let $X = \{x_1, \ldots, x_n\}$ and consider a subset of X, say $S = \{x_{i_1}, \ldots, x_{i_m}\}$.

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Let $X = \{x_1, ..., x_n\}$ and consider a subset of X, say $S = \{x_{i_1}, ..., x_{i_m}\}$.
Better: Let X be a finite set and consider a subset S of X. Let x be a member of S.

Unnecessary superscripts or subscripts

- If you refer to only one element of a set the subscript is superfluous.
- Do not write X_i in one place and then X_n somewhere else.
- Do not call a vector (x, y, z) in one place and then (x_1, x_2, x_3).

Superfluous symbols

Example

An integer z is called *even* if it is divisible by 2.

We do not need the symbol z.

Instead write:

An integer is called *even* if it is divisible by 2.

Example

This algorithm has $t = \log_2 n$ loops in the worst case.

The "$t =$ "is unnecessary unless you are using this sentence to define t.

Superfluous symbols

Theorem

$$\lim_{n \to \infty} a_n = \rho \geq 0.$$

We do not need ρ in the statement. The statement is easier to understand without ρ.

Theorem

$$\lim_{n \to \infty} a_n \geq 0.$$

Put

$$\text{Let } \rho = \lim_{n \to \infty} a_n.$$

into the proof.

Display

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- Display important formulas on a line by themselves, using $$ ...$$ or the `equation` environment if you want to refer to them.
- When not on a line by themselves, mathematical symbols should, if possible, be the same font size as the other characters in the sentence. The expression $(n(n+1)+1)/2$ is easier to read than $\frac{n(n+1)+1}{2}$. However avoid potentially ambiguous expressions such as $a/b+c$.

Small numbers

Small numbers (less than 10) should be spelled out when used as adjectives, but not when used as names.

Example

Each node at the input layer should have a maximum of one 1 and a minimum of one 0 for its entries.

Standard functions

Standard mathematical functions such as sin, cos, arctan, max, gcd etc. are set in roman type. Use the latex commands $\sin$, etc. which will take care of that for you. $*sin*$ will appear in italic.

Examples:

$\gcd(X+1, X^2-1)$; $\tan(\alpha)$; $\arccos(\pi/10)$; $\dim(V)$; $\det(M)$; $\exp(x)$; $\ln(y)$; $\min(S)$; $\limsup f(x)$.

Variable names

Multiple-letter variable names and computer code are set in courier type. Use {\tt ...}.

Example

In the following algorithm we use the data types `face` and `number`.

Confusing notation

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- If you write $(x_{11}, \ldots, x_{1n}, \ldots, x_{nn})$, it is not clear in which way this $n \times n$ - tuple is ordered.
- What is $2.3.10^5$? Is it 600000 or is it 230000 ? You can write $2.3 \cdot 10^5$ for the latter one (use the command \cdot)

When shall we use symbols?

Use Symbols when

- the idea would be too cumbersome to express in words;
- it is important to make a precise mathematical statement;
- necessary, e.g. if a symbol is shorter than an explanation in words or if the symbol will be used again.

Use Words

- if they don't take up much more space;
- if the symbol would be used only once or twice.

Avoid jargon

No mathematical jargon when it is easier to understand the concept in words.

Example

Let $Sk_{\text{int}}(X) = \overline{B_{\max_{\text{int}}}}$ where $B_{\max_{\text{int}}} = \{x, \exists r_x > 0, B(x, r_x) \subset X$ and if $B'(x', r') \subset X$ with $B(x, r_x) \subset B'(x', r')$ then $x' = x, r' = r_x\}$.

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Rather, give the reader some idea of what you are talking about:

We define the interior skeleton of an object X to be the closure of the set of the centres of all maximal balls that are included in X.

Sometimes symbols are more cumbersome than words!

Symbols

Explain in Words

what the symbols mean if that helps the reader. Stating things twice, in complementary ways, for difficult concepts is a good idea (and you can also illustrate the definition by en example).

Define!

Always ensure that every symbol or variable is defined (at least informally) before the first time you use it!

Definition:

Used to set forth the meaning of a word, phrase or expression.

Ask 3 questions: **why?, where?** and **how?.**

Can be formal or informal.

Why?

Is it necessary to make the definition?

not, if it is only ever used once or twice.

not, if inappropriate (unnecessary).

An unnecessary definition:

A is *contractive* with respect to the norm $||\cdot||$ if $||A|| < 1$.

It is unnecessary because you could use

$$\text{“}A \text{ with } ||A|| < 1\text{”}$$

everytime you want to use

$$\text{“contractive } A\text{”}.$$

Where?

- Define a new term shortly before it is first used.
- Don't put many definitions at the beginning of your work.
- Remind your readers of terms that have not been used for several pages.

How?

- As short and precise as possible.
- Consistent with related definitions.
- Emphasise basic idea or fundamental property of the term you are defining.
- The term to be defined is highlighted by setting it in *italics*.

Example of a formal definition

Definition 1. If A and B are sets, the *intersection* of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Proposition 2. *Let Y be a set with $Y \subseteq A$ and $Y \subseteq B$. Then $Y \subseteq A \cap B$.*

Proof.

For each $y \in Y$ it follows that $y \in A$ and $y \in B$ as $Y \subseteq A$ and $Y \subseteq B$. Hence $y \in A \cap B$. Therefore, $Y \subseteq A \cap B$. □

Example of an informal definition

For two sets A and B the *intersection* $A \cap B$ is the set of all elements which belong to both A and B. If Y is any subset of A and also of B, then Y is also a subset of $A \cap B$ because every element of Y lies in both A and B and hence in the intersection.

It's not necessary to follow these concepts formally as they are quite straightforward.

Theorem or Lemma?

Theorem:

- major result.
- usually is of interest outside the context of the current text.

Lemma:

- auxiliary result.
- a step towards proving theorem.

However, there are some famous results which are called lemma.

The statement of the theorem should usually be self-contained, not depending on the assumptions of the previous text.

Other results

Proposition:

Used less frequently. Describes a minor result.

Corollary:

An immediate consequence of a previous result and easy to prove.

Conjecture:

A statement that has not been proved. Usually the author of a conjecture has strong evidence that it might be correct.

Hypothesis:

An assumption on which any further reasoning is based. Usually the statement of a theorem, lemma, proposition or corollary contains a hypothesis.

Proofs

Proofs are given to allow a reader to verify results. When writing proofs keep two types of readers in mind:

- those who are not interested in the minute details of the proof but would like to understand the basic ideas.
- those who want to understand every detail with as little effort as possible.

It is helpful to outline the structure of the proof and the difficulty of the steps involved.

Examples

- What is the purpose of this example?
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Examples **before** a result or a definition can:

- strike connection to well known concept;
- assist in finding abstraction of concept;
- illustrate idea used later very clearly;
- motivate result/definition;
- motivate proof.

Examples

Examples **after** a result or a definition can:

- illustrate what was just discussed;
- give experience with a newly defined term;
- highlight difficulties;
- provide a thought model for reader when using a newly defined term or result.

Motivate

Try to motivate the reader to what follows.

Example

Using xxx, we can now show the following result.

Theorem 1. yyy

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Using xxx, we can now show the following result.

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Note that the statement right before a theorem or algorithm must be a complete sentence.

Bad example

We now prove the following

Theorem 1. yyy

Motivate

Try to motivate the reader to what follows.

Avoid a list of Theorems/Lemmas with no explanation in between.

Keep the reader in mind as you are writing. What does the reader know so far? What does the reader expect next, and why?

Conclusion

- It is definitely a long learning process to get the right balance of motivation/technical details/examples; not too long/not too short.

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- Practice a lot, get feedback from your supervisor/your peers.
- No ambiguity, easiness to read, conciseness, flow are some important things to keep in mind.