1. What is the cardinality of the following sets?

(a) The power set of the set \( \{ \emptyset, \{\emptyset\} \} \)

**Solution:** The cardinality of the original set is 2, so the cardinality of the power set is \( 2^2 \), or 4.

(b) The power set of the empty set

**Solution:** The cardinality of the empty set is 0, so the cardinality of the power set is \( 2^0 \), or 1. (The power set of the empty set is: \( \{ \emptyset \} \).

(c) The power set of all functions from boolean values (i.e. the set \( \{true, false\} \)) to boolean values

**Solution:**
Let us call the set of functions from booleans to boolean \( S \). For each element in the domain, (\( \{true, false\} \)), a function \( f \) must have a value drawn from the codomain (also (\( \{true, false\} \)). So the total number of functions is \( 2 \times 2 \), or 4.
The cardinality of the power set of \( S \) is \( 2^4 \), or 16.

2. Show that the following two sets have the same cardinality!
(a) The set of natural numbers including zero and the set of integers, i.e. show
\[ |\mathbb{N}_0| = |\mathbb{Z}|. \]

\textbf{Solution:} In order to show that the two sets have the same cardinality we need to find a bijection between the two sets: Define the following function:

\[ f : \mathbb{N}_0 \to \mathbb{Z}, \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \]

This function is well defined as every element of the domain maps to at least one element of the co-domain, i.e. we have define a function. Now we need to show that this function is a bijection. \( f \) is surjective, because for each integer \( z \in \mathbb{Z} \) we can find a positive integer \( n \in \mathbb{N}_0 \) such that \( f(n) = z \). There are two possible cases: Either \( z \) is negative or not. If \( z \) is not negative, then \( n := 2 \cdot z \) is in \( \mathbb{N}_0 \) and \( f(n) = f(2 \cdot z) = \frac{2z}{2} = z \).

If \( z \) is negative, we choose \( n \) to be \( n := (-2 \cdot z) - 1 \). Since \( z \) is negative, \( n \) is a natural number and \( f(n) = f((-2 \cdot z) - 1) = -\frac{(-2z) - 1 + 1}{2} = z \). Therefore, \( f \) is surjective.

Let us now show that \( f \) is injective: Let \( n_1, n_2 \in \mathbb{N}_0 \) and \( f(n_1) = f(n_2) \). If both are even, then

\[ f(n_1) = f(n_2) \]
\[ \iff \frac{n_1}{2} = \frac{n_2}{2} \]
\[ \iff n_1 = n_2. \]

Similarly, we can show the same in the case that both are odd.

If \( n_1 \) is even and \( n_2 \) is odd (or vice-versa) then

\[ f(n_1) = f(n_2) \]
\[ \iff \frac{n_1}{2} = -\frac{n_2 + 1}{2} \]
\[ \iff n_1 = -(n_2 + 1) \]

which means that \( n_1 \) or \( n_2 \) must be negative, which cannot be the case since they are both positive integers. Therefore, \( f \) is injective. We have shown that \( f \) is a bijection between the two sets, therefore the \( |\mathbb{N}_0| = |\mathbb{Z}|. \)
(b) Let $A$ be a finite set. Show that the set of all functions $f: A \to \{0, 1\}$ has the same cardinality as the power set $\mathcal{P}(A)$.

**Solution:** We do not give a formal proof here but we want to give a brief explanation instead:
The power set $\mathcal{P}(A)$ is the set of all possible subsets of $A$. In order to build these subsets, we need to decide each time whether a specific element of $A$ is part of a subset or not. Then we count how many possibilities there are to put elements of $A$ together and build different subsets of $A$. Choosing whether an element of $A$ is in a subset is equivalent to a function that maps this element either to 1 or 0 (1 meaning that this element is in the subset and 0 meaning it is not). Therefore, each subset can be linked to a function that maps exactly those elements in the subset to 1 and all others to 0. These functions are all different for each subset there is one of these funtions and the other way around. This linkage can be defined formally in the following way: Let $\mathcal{M}$ be the set of all functions $f : A \to \{0, 1\}$.

$$g : \mathcal{P}(A) \to \mathcal{M}; \quad S \mapsto f \in \mathcal{M} \text{ with } f(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases}$$

As in part 2a, we can now show that this function is a bijection.

3. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

(a) How many functions $f : A \to B$ are there?

(b) How many functions $f : A \to B$ are injective, how many surjective?

(c) How many functions $f : A \to B$ are bijective?

**Solution:**

(a) There are 2 choices for $f(1)$, 2 choices for $f(2)$ and 2 choices for $f(3)$, so there are $2 \times 2 \times 2 = 8$ such functions.

(b) Since $A$ has size 3 and $B$ has size 2, there are no such injections. To count surjections, it is easier to count the “non-surjections”.

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For $f$ not to be a surjection, it has to map every element of $A$ to the same element of $B$, either $a$ or $b$. Hence there are 2 non-surjections. The total number of functions, as seen in Part (a), is 8. Thus there are 6 surjections.

(c) If there existed a bijection from $A$ to $B$, we would have $|A| = |B|$. So there are no bijections.

4. Show that the union of two countable sets is countable.

**Solution:** If the two sets are finite, it is obvious. If one is finite and the other infinite countable, make a list by first putting the finite set in any order then the infinite set, which we know can be ordered in a list since it is countable.

Now assume both sets $A$ and $B$ are infinite countable. We know we can order both sets $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Now we can order $A \cup B$ the following way:

\[
\begin{align*}
& a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10} \quad \ldots \\
& b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5 \quad b_6 \quad b_7 \quad b_8 \quad b_9 \quad b_{10} \quad \ldots
\end{align*}
\]

so $A \cup B$ is infinite countable.

5. Which of the following sets are *countable*? Justify your answer.

(a) The set $\mathcal{P}(\mathbb{N}_{\geq 0})$, the power set of the positive natural numbers.

(b) The set $S$ of all functions $f : \{0, 1\} \to \mathbb{N}_{\geq 0}$.

(c) The set $T$ of all functions $f : \mathbb{N}_{\geq 0} \to \{0, 1\}$.

(d) The set $A \times A$, where $A$ is an infinite countable set.

(e) The *successor* function on the natural numbers, which maps each natural number to the subsequent natural number.
Solution:

(a) The set \( \mathcal{P}(\mathbb{N}_{\geq 0}) \) is uncountable because the power set of a set always has greater cardinality than the set.

(b) The set \( S \) of all functions \( f : \{0, 1\} \to \mathbb{N}_{\geq 0} \) is countable because it has the same cardinality as \( \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \), which we have seen in lectures is countable. To prove that we will exhibit a bijection between \( S \) and \( \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \):

\[
g : S \to \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \text{ defined by } g(f) = (f(0), f(1)).
\]

- \( g \) is injective since if \( g(f_1) = g(f_2) \) then \( (f_1(0), f_1(1)) = (f_2(0), f_2(1)) \) so \( f_1(0) = f_2(0) \) and \( f_1(1) = f_2(1) \), which means that \( f_1 \) and \( f_2 \) are the same function.

- \( g \) is surjective since \((a, b) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}\) is \( g(f) \) for the function \( f \) mapping 0 to \( \text{a} \) and 1 to \( b \).

(c) The set \( T \) of all functions \( f : \mathbb{N}_{\geq 0} \to \{0, 1\} \) is uncountable because it is the same size as \( \mathcal{P}(\mathbb{N}_{\geq 0}) \). To prove that we will exhibit a bijection between \( T \) and \( \mathcal{P}(\mathbb{N}_{\geq 0}) \):

\[
g : T \to \mathcal{P}(\mathbb{N}_{\geq 0}) \text{ defined by } g(f) = \{ n \in \mathbb{N}_{\geq 0} | f(n) = 1 \}.
\]

- \( g \) is injective since if \( g(f_1) = g(f_2) \) then \( \{ n \in \mathbb{N}_{\geq 0} | f_1(n) = 1 \} = \{ n \in \mathbb{N}_{\geq 0} | f_2(n) = 1 \} \) so \( f_1(n) = 1 \) exactly whenever \( f_2(n) = 1 \). It follows \( f_1(n) \neq 1 \) (so \( f_1(n) = 0 \)) exactly (You can refer to question 6(c).) whenever \( f_2(n) \neq 1 \) (so \( f_2(n) = 0 \)), which means that \( f_1 \) and \( f_2 \) are the same function.

- \( g \) is surjective since \( C \in \mathcal{P}(\mathbb{N}_{\geq 0}) \) is \( g(f) \) for the function \( f \) mapping every element in \( C \) to 1 and everything else to 0.

(d) The set \( A \times A \) is countable, by a zig-zag argument similar to the one we saw in lectures for \( \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \).

(Note that that is perfectly allowable to ask the size of a function, in set theory – since a function is just a special type of relation, and a relation is just a set of tuples.)
SOLUTION: (alternative solution for (b)):
Any function \( f \) from the set \( \{0, 1\} \) to the positive natural numbers will be of the form:
\[
f(n) = \begin{cases} x & \text{when } n = 0, \\ y & \text{when } n = 1, 
\end{cases}
\]
where \( x \) and \( y \) are two positive natural numbers.
Let us give these functions names - \( f_{x, y} \) will be the name of the function for which \( f(0) = x \) and \( f(1) = y \).
We can now put the names of this functions in a grid, as we have done in lectures with pairs of integers:

\[
\begin{array}{ccc}
f_{1,1} & f_{1,2} & f_{1,3} \\
f_{2,1} & f_{2,2} & f_{2,3} \\
f_{3,1} & f_{3,2} & f_{3,3} \\
\vdots & & \vdots \\
\end{array}
\]

We can then exhibit a bijection between these and the natural numbers, by enumerating them in a “zig-zag” fashion (refer to to lecture notes for the function).
Therefore, the number of functions is countable.

SOLUTION: Alternative solution for (c):

- We can show that the set of functions \( f : \mathbb{N}_{\geq 0} \to \{0, 1\} \) has the same cardinality as another set we have shown is uncountable: the set of infinite sequences of the numbers 0 and 1.
- Consider some function \( f_i \) in the set of such functions. For every element \( n \) in \( \mathbb{N}_{\geq 0} \), it must map that \( n \) either to 0 or to 1. Which is exactly the definition of an infinite but countable sequence of 0s and 1s.
- So each function \( f_i \) is equivalent to a countable infinite sequence of 0s and 1s.
- Now, the set of all functions \( f : \mathbb{N}_{\geq 0} \to \{0, 1\} \) means that we’ll have every possible such function; which is exactly equivalent to the set of all possible sequences of 0s and 1s.
- And we saw that, by Cantor’s Diagonal Argument, it’s impossible to make a list of the set of all possible 0-1-sequences; therefore the set of them is uncountable.
Therefore, the set of functions \( f : \mathbb{N}_{\geq 0} \to \{0, 1\} \) is also uncountable. \( \square \)

6. For each of the following sets, state whether it is finite, countably infinite, or uncountable.

(a) The power set of the real numbers, \( \mathbb{R} \).

Solution: It is uncountable. We know that the set of real numbers, \( \mathbb{R} \), is uncountable. And we know that the cardinality of the power set of any set is strictly greater than the cardinality of that set (Cantor’s Theorem). Therefore the cardinality of the power set of the reals, \( \mathcal{P}(\mathbb{R}) \), must be uncountable as well.

(b) The set difference of the real numbers and the integers (i.e., \( \mathbb{R} \setminus \mathbb{Z} \)).

Solution:

It is uncountable. One argument: Consider just the set of real numbers \( r \), in the interval \( 0 < r < 1 \). We could write those numbers in binary (e.g. 0.0101010..., and so on), and each one of those would be an infinite sequence of binary digits. We have from Cantor’s diagonal argument that the set of binary infinite sequences is uncountable; and we also know that the superset of an uncountable set is uncountable; so it follows that \( \mathbb{R} \setminus \mathbb{Z} \) must be uncountable.

A second argument: Let \( S \) be the set \( \mathbb{R} \setminus \mathbb{Z} \). We will prove by contradiction that it is countable. Assume the contrary, that it is countable. We know that \( \mathbb{Z} \) is countable, and we know that the union of two countable sets is also countable; therefore it follows that \( S \cup \mathbb{Z} \) is countable. But \( S \cup \mathbb{Z} \) is exactly \( \mathbb{R} \), and we are given that \( \mathbb{R} \) is uncountable. We have derived that \( \mathbb{R} \) both is and is not uncountable, which is a contradiction. Therefore our assumption (that \( S \) is countable) must be false.

(c) The set of even integers greater than \( 10^{100} \).
Solution: This is countable. The set of integers, \( \mathbb{Z} \), is countable, and we know that the subset of a countable set is also countable. Therefore the set of even integers is countable, and so is the set of even integers greater than 100.

7. Prove that \( |A \cup B| = |A| + |B| - |A \cap B| \), when \( A \) and \( B \) are finite sets.

Solution: We have that \( A \cup B \) is the disjoint union of \( A - B \), \( B - A \) and \( A \cap B \), so \( |A \cup B| = |A - B| + |B - A| + |A \cap B| \). Now \( A \) is the disjoint union of \( A - B \) and \( A \cap B \), so \( |A| = |A - B| + |A \cap B| \). Similarly \( |B| = |B - A| + |A \cap B| \).

Thus \( |A - B| = (|A| - |A \cap B|) + (|B| - |A \cap B|) + |A \cap B| = |A| + |B| - |A \cap B| \).

Solution: (alternative)
Divide \( A \cup B \) into two disjoint parts:
\[
A \cup B = A \cup (B - A).
\]
We know they’re disjoint, so the cardinality is
\[
|A \cup B| = |A| + |B - A|.
\]
Now let’s work out the cardinality of \( B - A \).
\[
B - A = B - (A \cap B),
\]
and we know they are disjoint, too, so
\[
|B - A| = |B| - |A \cap B|.
\]
So the cardinality of the whole expression is
\[
|A \cup B| = |A| + |B| - |A \cap B|.
\]

8. There is a smallest legal Java program. It looks something like:
\[
\text{public class A\{public static void main(String\[] a)\{\}\}}
\]
What is the cardinality of the set of all legal Java programs, \( J \)?

There is also a set of functions from \( \mathbb{N} \) to \( \{T, F\} \) - call it \( P \). What is the cardinality of this set?

How do the cardinalities of \( J \) and \( P \) compare – is one larger than the other? If so, what do you deduce from this?