1 Revision questions (for self-study)

1. Prove, using induction, that for any integer \( n \), \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).
   (That is, \( 1 + 2 + ... + n = n(n+1)/2 \).)

2. State whether the following statement is true or false, and prove (or disprove) it:
   If \( A \subseteq B \) and \( C \subseteq D \), then \( A \times C \subseteq B \times D \).

3. Phyllis Filum is designing a file-system for her new operating system, Finux, and is modelling it mathematically. She is using the symbol \( R \) to represent the relation “is a child of”, holding between items in the file-system (files or directories). Thus, for two directories \( a \) and \( b \), \( aRb \) means that \( b \) is directly contained within \( a \).
   
   Consider the properties (of being reflexive, transitive, symmetric or antisymmetric) which a relation could have. Which properties does \( R \) possess – do you have enough information to answer this? If not, what further information would you need about how Phyllis’s file-system works?

2 Functions

1. The following directed graph represents a relation on a set of integers. Is it a function?

2. Of the possible types of relation (one-to-many, many-to-one, one-to-one, many-to-many), which can be functions?
3. Which of the following functions are injective, which surjective, and which bijective?

(a) \( f_1 : \mathbb{N} \to \mathbb{N} \) where \( f_1(n) = 2n \).
(b) \( f_2 : \mathbb{N} \to E \) where \( f_2(n) = 2n \) (recall \( E \) is the set of natural even numbers).
(c) \( f_3 : \mathbb{N} \to \mathbb{N} \) where \( f_3(n) = n/2 \) if \( n \) is even and \( f_3(n) = (n + 1)/2 \) otherwise.
(d) \( f_4 : \mathbb{R} \to \mathbb{R} \) where \( f_4(x) = 2x \).
(e) \( f_5 : \mathbb{R} \to \mathbb{R}_{\geq 0} \) where \( f_5(x) = x^2 \).
(f) \( f_6 : \mathbb{R} \to \mathbb{R}_{\geq 0} \) where \( f_6(x) = x^3 - 4x \).

4. Which of the following functions are injective, which surjective, and which bijective?

a. The function which maps from any prime number to the next largest prime number.
b. The function which maps from any natural number, \( n \in \mathbb{N}_p \), to its set of prime factors.
c. The function which, for a string representing a valid Java program, maps to the same program but with all comments and whitespace removed.
d. The function which, for a string representing a valid Java program, maps to the set of all possible strings it could emit on \texttt{System.out}.

5. Draw diagrams of the graphs for the following functions:

(a) On the set of weekdays, \{\textit{Monday, Tuesday, Wednesday, Thursday, Friday}\}, the function which maps from a weekday to the following weekday. (Monday is the weekday which follows Friday.)
   Is it injective? Surjective? Bijective?
(b) On the set of weekdays, \{\textit{Monday, Tuesday, Wednesday, Thursday, Friday}\}, the function which maps from every weekday to Friday.
   Is it injective? Surjective? Bijective?

6. Are the following relations functions?

(a) The relation “father of”, holding between people.
(b) The relation “is greater than”, holding between integers.
(c) The relation “is 1 greater than”, holding between integers.
(d) The relation “is 1 greater than”, holding between natural numbers (\( \mathbb{N}_{\geq 0} \)).

7. (a) Is it possible to have a surjective function from \( \mathbb{N}_{\geq 0} \) to \( \mathbb{Z} \)? If so, describe one.
   (b) Is it possible to have a bijective function from \( \mathbb{Z} \) to \( \mathbb{N}_{\geq 0} \)? If so, describe one.

3 Extension questions

1. The \textit{division} function over the integers maps from pairs of integers, \((m_1, m_2)\), to some other integer, call it \( n \). What sort of function is this? Is it injective, surjective, bijective? Is there anything unusual about it?

2. State whether the following statement is true or false, and prove (or disprove) it:
   There is a bijection between the set of all valid Java programs and the natural numbers (\( \mathbb{N}_{\geq 0} \)).
3. Composition is the process of applying one function to the output of another, and is written using a circle (◦); if a function $f$ can be applied to the output of a function $g$, then we can write the function $f(g(x))$ as $f \circ g$.

Prove (or disprove) that if $f$ is a bijection, and $g$ is a bijection, then $f \circ g$ is also a bijection.

4. The following bipartite graph represents a relation between the empty set ($\emptyset$, on the left), and the strictly positive natural numbers ($\mathbb{N}_{>0}$, on the right). Is it a function?

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∅  N
  ↓  ↓
  1  2
  3
...
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5. Functions can take one argument (for instance, the function $f(n) = 2n$), or they can take multiple arguments (for instance, the addition function, $+ : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$). They can also take zero arguments; for instance, consider the function $f() = 3$. What are some other examples of functions which take zero arguments? What are their domain and codomain? Are they injective, surjective, bijective? Do they remind you of any of the sorts of things we saw in predicate logic?

6. Is it possible to have a function whose domain is the empty set? Whose codomain is the empty set? Both at once? What will such functions look like – will they be injective, surjective, bijective?