1. If $A = \{1, 2, 3\}$ then write down $\mathcal{P}(A)$.

**Solution:** Construct the power set for any set $A$ of size $n$ by adding elements as follows:

- The empty set is an element of $\mathcal{P}(A)$. (1)
- Include a singleton (single element) set for each element of $A$ ($n$)
- Include 2-element sets for each distinct pair of elements from $A$ ($n$ choose 2)
- Include 3-elements sets... and so on ($n$ choose 3 etc)
- Include the set $A$. (1)

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Challenge: use induction and the construction above to prove that the size of the powerset of any set $A$ with size $n$ is the $2^n$.

2. If $A = \{\emptyset, 1, 2, \{1, 2\}\}$, then determine which of the following are true statements.

**Solution:** First, write down (some of) $\mathcal{P}(A)$. This set has $2^4 = 16$ elements, that is, $1 + 4 + 6 + 4 + 1 = 16$.

$$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{2\}, \{\{1, 2\}\}, \text{ all subsets of 2 elements...}, \text{ all subsets of 3 elements...}, \{\emptyset, 1, 2, \{1, 2\}\}\}.$$ 

(a) $1 \in A$

**Solution:** True. 1 is one of the four members of $A$

(b) $\{1\} \in A$

**Solution:** False. This one-element set is not a member of $A$

(c) $\{1\} \in \mathcal{P}(A)$

**Solution:** True. This is a subset of $A$ and therefore a member of $\mathcal{P}(A)$

(d) $\{1\} \subseteq \mathcal{P}(A)$
Solution: False, the members of $\mathcal{P}(A)$ are subsets not atoms

(e) $\emptyset \in A$

Solution: True, this set does contain $\emptyset$ as one of its elements

(f) $\emptyset \subseteq A$

Solution: True, the empty set is a subset of any set

(g) $\emptyset \in \mathcal{P}(A)$

Solution: True, subsets of $A$ are members of $\mathcal{P}(A)$ and $\emptyset$ is a subset of any set.

(h) $\{1, 2\} \in A$

Solution: True, this set is in $A$

(i) $\{1, 2\} \subseteq A$

Solution: True, this is one of the subsets of size 2 since $A$ contains both 1 and 2 as elements

(j) $\{\emptyset, 1\} \in A$

Solution: False, this set is not a member of $A$

3. Draw a Venn Diagram of three sets $A$, $B$, $C$ and shade in the regions corresponding to the sets

(a) $A \cup (B \cap C)$
(b) $A \cap (B \cup C)$
(c) $(A \backslash B) \backslash C$

Solution:

\[
\begin{array}{ccc}
A \cup (B \cap C) & A \cap (B \cup C) & (A \backslash B) \backslash C \\
\includegraphics[width=.3\textwidth]{venn_aubcapc} & \includegraphics[width=.3\textwidth]{venn_aubc} & \includegraphics[width=.3\textwidth]{venn_abbc} \\
A & B & C \\
\end{array}
\]
4. Is it true that \( A = B \) whenever \( A \cup C = B \cup C \)?

(If you believe that the statement is true, then you must give a proof of this fact, while if you believe that it is not always true, then you must give a counterexample, which is an explicit choice of \( A, B \) and \( C \) for which the statement does not hold.)

**Solution:** The statement is false. Suppose that \( A = \{0\} \), \( C = \{1\} \) and \( B = \{0, 1\} \). Then \( A \cup C = \{0, 1\} = B \cup C \), but clearly \( A \neq B \).

5. If \( A, B \) are both subsets of a universal set \( U \) then prove that
\[
A \cap B = A \cup B
\]

Hint: prove that if \( x \) is a member of the LHS set then it also a member of the RHS set and if \( x \) is a member of the RHS set it is also a member of the LHS set.

**Solution:**
We need to show that
\[
A \cap B \subseteq A \cup B
\]
and
\[
A \cup B \subseteq A \cap B
\]

To prove the first:
Let \( x \in A \cap B \)
Then \( x \notin A \cap B \)
So \( x \) is not in both \( A \) and \( B \)
So \( x \) is not in \( A \) or \( x \) is not in \( B \)
So \( x \in \overline{A} \) or \( x \in \overline{B} \)
Hence \( x \in \overline{A} \cup \overline{B} \)
Therefore \( A \cap B \subseteq \overline{A} \cup \overline{B} \)

To prove the second:
Let \( x \in \overline{A} \cup \overline{B} \)
Then \( x \in \overline{A} \) or \( x \in \overline{B} \)
So \( x \notin A \) or \( x \notin B \)
So \( x \) is not in \( A \cap B \)
So \( x \in \overline{A} \cap \overline{B} \)
Therefore \( \overline{A} \cup \overline{B} \subseteq \overline{A} \cap \overline{B} \)

6. For each of the following binary relations on the natural numbers (i.e. \( R \subseteq \mathbb{N} \times \mathbb{N} \)) state whether the relation is one to many, one to one, many to many or many to one. Give a brief reason for each answer.

(a) \( \{(1, 2), (1, 4), (1, 6), (2, 3), (4, 3)\} \)
(b) \( \{(9, 7), (6, 5), (3, 6), (8, 5)\} \)
(c) \( \{(12, 5), (8, 4), (6, 3), (7, 12)\} \)
(d) \( \{(2, 7), (8, 4), (2, 5), (7, 6), (10, 1)\} \)
Solution:

(a) Many to many because 1 maps to 2, 4 and 6, and 2 and 4 both map to 3.
(b) Many to one because both 6 and 8 map to 5, but no y maps to the same x.
(c) One to one because each x maps to a unique y, and each y has a unique x.
(d) One to many because 2 maps to 7 and 5, but each y has a unique x.

7. Characterise (as 1-1, 1-M, M-1, M-M) each of the following binary relations R on the given sets. Give a brief reason for each answer.

(a) \( R \subseteq \mathbb{N} \times \mathbb{N} \) where \( R = \{(x, y) \mid x = y + 1\} \)
(b) \( R \subseteq \mathbb{R} \times \mathbb{R} \) where \( R = \{(x, y) \mid x = 5\} \)
(c) \( R \subseteq \text{females} \times \text{females} \) where \( R = \{(x, y) \mid \text{daughterof}(x, y)\} \)
(d) \( R \subseteq \text{people} \times \text{people} \) where \( R = \{(x, y) \mid \text{daughterof}(x, y)\} \)

Solution:

(a) 1-1. No two x values map to the same y and no two y values have the same x.
(b) One to many. One x maps to many different y (e.g. (5,5), (5,6), (5,1) etc).
(c) Many to one. Assuming that, \( \text{daughterof}(x, y) \) means \( x \) is a daughter of \( y \) One female (mother) can have many daughters, but no daughter has more than one mother.
(d) Many to many. One parent (male of female) can have many daughters. And each daughter has 2 parents.

8. Which properties (R,T,S,A) does each of the following binary relation R on the set \( A = \{0, 1, 2, 4, 6\} \) have? Give reasons for your answer.

\[ R = \{(0, 0), (1, 1), (2, 2), (4, 4), (6, 6), (0, 1), (1, 2), (2, 4), (4, 6)\} \]

Solution:

R is reflexive, because \((a, a) \in R\) for all \( a \in A \).
R is not symmetric, because \((0, 1) \in R\) but \((1, 0) \notin R\)
R is not transitive because \((0, 1) \in R \land (1, 2) \in R\) but \((0, 2) \notin R\).
R is anti-symmetric because the only pairs where \((a, b) \in R\) and \((b, a) \in R\) have \( a = b \), that is \((0, 0)\) etc.

9. Which properties does each of the following binary relation on the set \( \mathbb{N} \) have?

\[ R = \{(x, y) \mid \text{even}(x \times y)\} \]

(That is, \( x \) times \( y \) is an even number)
Solution:

R is not reflexive, because (3, 3) \(\notin R\) since 9 is not even.
R is symmetric, because \(x \times y = y \times x\), so if one is even, then so is the other.
R is not transitive because (3, 2) \(\in R\) \(\land\) (2, 5) \(\in R\) but (3, 5) \(\notin R\) (6 and 10 are even but 15 is not).
R is not anti-symmetric (3, 2) \(\in R\) \(\land\) (2, 3) \(\in R\) but 3 \(\neq 2\).

10. Consider the relation \(\subseteq\) defined on the power set \(\mathcal{P}\{1, 2, 3\}\) (recall that the power set of \(A\) is the set of all subsets of \(A\)). Give reasons to support your answers for each of the following.

(a) Is \(\subseteq\) reflexive?

(b) Is \(\subseteq\) transitive?

(c) Is \(\subseteq\) symmetric?

(d) Is \(\subseteq\) antisymmetric?

Solution:

(a) Is \(\subseteq\) reflexive? Yes since every set is a subset of itself.

(b) Is \(\subseteq\) transitive? Yes since if \(A\) is contained in \(B\) and \(B\) in \(C\), then \(A\) is also in \(C\).

(c) Is \(\subseteq\) symmetric? No because \(\emptyset \subseteq \{0, 1\}\) but \(\{0, 1\} \nsubseteq \emptyset\)

(d) Is \(\subseteq\) antisymmetric? Yes, since whenever \(A \subseteq B\) and \(B \subseteq A\) then the sets must be equal by the definition of set equality.

11. Draw the following relations as bipartite graphs (ie. a graph showing edges between only a,b pairs in the relation - see the course notes for details)

(a) Your timetabled classes and day of the week e.g. \(\text{cits2211,Mon}\) and \(\text{cits2211,Thu}\) are in the relation.

(b) the relation divides on the set \(\{1, 2, 3, \ldots, 10\}\). We say \(a\) divides \(b\) if and only if \(b\) is a multiple of \(a\). For example, \((2, 6) \in \text{divides}\) but \((2, 5) \notin \text{divides}\).

Solution: This will be done in class.

12. (challenge) Relations can be used to model preferences using \(x \geq y\) to mean that \(x\) is “at least as good as” \(y\) or I prefer \(y\) over \(x\).

Consider 1000 cups of coffee, numbered \(C_0, C_1, C_2, \ldots\) up to \(C_{999}\). Cup \(C_0\) contains no sugar, cup \(C_1\) one grain of sugar, cup \(C_2\) two grains etc. Since one cannot taste the difference between \(C_{999}\) and \(C_{998}\), they are equally good (of equal value), \(C_{999} \geq C_{998}\) or \(C_{999}\) is no worse than \(C_{998}\). For the same reason, we have \(C_{998} \geq C_{997}\), etc. all the way up to \(C_1 \geq C_0\). Since preference is transitive, we should have \(C_{999} \geq C_0\) which means that \(C_{999}\) is no worse than \(C_0\). But clearly \(C_0\) is worse than \(C_{999}\). This contradicts transitivity of indifference, and therefore also transitivity of weak preference.

Suggest some possible ways of resolving this paradox?
SOLUTION: This is one version of a famous paradox called the Sorites paradox. See the Stanford Encyclopedia of Philosophy for some possible resolutions of the paradox. http://plato.stanford.edu/entries/sorites-paradox