1. Prove that the sum of the first \( n \) odd numbers is equal to \( n^2 \).
   That is, prove that for all \( n \geq 1 \),

\[
1 + 3 + \cdots + (2n - 1) = n^2
\]

or

\[
\sum_{i=1}^{n} (2i - 1) = n^2
\]

(This is an elementary “crank the handle” or “plain vanilla” induction proof, so you should focus on formatting it as correctly as possible.)

2. Prove that for any \( n \geq 1 \), the value \( 11^n + 4 \) is divisible by 5.

3. Prove that any class of 18 or more students can be assembled into teams of size 4 or 7.

For more practice questions, see Chapter 5 of Mathematics for Computer Science, and the resources listed on the CITS2211 website.

Challenge Questions

1. Consider a \( 2^n \times 2^n \) chessboard, with a single square removed from the top-right corner. Show that any such chessboard can be completely covered by L-shaped tiles as shown in the diagram.

   (This is a more interesting induction proof as it is not just a simple statement about integers; however it is quite straightforward, and only needs weak induction.)
2. How far can you generalize the proof of Theorem 1.8.1 (below) that \( \sqrt{p} \) is irrational? For example, how about \( \sqrt{3} \)? Can you generalise this with an inductive hypothesis?

Source: Problem 1.13. Mathematics for Computer Science

**Example**

We’ll prove by contradiction that \( \sqrt{2} \) is irrational. Remember that a number is *rational* if it is equal to a ratio of integers—for example, \( 3.5 = 7/2 \) and \( 0.1111\cdots = 1/9 \) are rational numbers.

**Theorem 1.8.1.** \( \sqrt{2} \) is irrational.

**Proof.** We use proof by contradiction. Suppose the claim is false, and \( \sqrt{2} \) is rational. Then we can write \( \sqrt{2} \) as a fraction \( n/d \) in *lowest terms.*

Squaring both sides gives \( 2 = n^2/d^2 \) and so \( 2d^2 = n^2 \). This implies that \( n \) is a multiple of 2 (see Problems 1.11 and 1.12). Therefore \( n^2 \) must be a multiple of 4. But since \( 2d^2 = n^2 \), we know \( 2d^2 \) is a multiple of 4 and so \( d^2 \) is a multiple of 2. This implies that \( d \) is a multiple of 2.

So, the numerator and denominator have 2 as a common factor, which contradicts the fact that \( n/d \) is in lowest terms. Thus, \( \sqrt{2} \) must be irrational.