1. Prove that the sum of the first $n$ odd numbers is equal to $n^2$.

That is, prove that for all $n \geq 1$,

$$1 + 3 + \cdots + (2n - 1) = n^2$$

or

$$\sum_{i=1}^{n} (2i - 1) = n^2$$

(This is an elementary “crank the handle” or “plain vanilla” induction proof, so you should focus on formatting it as correctly as possible.)

**Solution:** Let $P(n)$ denote the predicate

$$\forall n \geq 1. \ 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

which can also be written less ambiguously as $\sum_{i=1}^{n} (2i - 1) = n^2$.

**Base Case:** For $n = 1$, the left-hand side is a “sum” of one number, so is equal to 1 and the right hand side is $1^2 = 1$, and so $P(1)$ is true.

**Inductive Step:** Suppose that $P(k)$ holds, that is, the result is true for $n = k$. We wish to prove that $P(k + 1)$ is true, so write down the left-hand side

LHS of $P(k+1)$ is $1+3+5+\ldots+(2k-1)+(2(k+1)-1) = 1+3+5+\ldots+(2k-1)+(2k+1)$

By the induction hypothesis this can be re-written as $k^2 + (2k + 1) = (k + 1)^2$ which is the RHS of $P(k + 1)$ so $P(k + 1)$ holds.

Therefore, by the Principle of Mathematical Induction (PMI) we conclude that $P(n)$ holds for all $n \geq 0$.

QED

2. Prove that for any $n \geq 1$, the value $11^n + 4$ is divisible by 5.

**Solution:** We use induction to prove the statement $\forall n. \ n \geq 1 \rightarrow P(n)$ where $P(n)$ is

$$5 \mid (11^n + 4)$$

which can also be written

$$\forall n. \ n \geq 1 \rightarrow \exists a. \ 11^n + 4 = 5a$$
The base case If $n = 1$, then $11^1 + 4 = 15$, which is divisible by 5, so $P(1)$ is true.

The inductive step Assume that $P(k)$ is true. By the inductive hypothesis there is some $a$ so that $11^k + 4 = 5a$ and $11^k = 5a - 4$. We also know that $a > 1$ since $5a = 11^k + 4 \geq 15$ so $a \geq 3$ for any $k$. Then do some arithmetic as follows:

$$LHS = 11^{k+1} + 4 = 11(11^k) + 4 = 11(5a - 4) + 4 = 55a - 44 + 4 = 55a - 40 = 5(11a - 8)$$

which shows that $11^{k+1} + 4$ is divisible by 5, and so we have proved $P(k + 1)$

Therefore $P(k) \rightarrow P(k + 1)$ and so by the PMI, $P(n)$ holds for all $n \geq 1$.

QED

3. Prove that any class of 18 or more students can be assembled into teams of size 4 or 7.

**Solution:** We use induction to prove $P(n) = \exists a, b. a \geq 0 \land b \geq 0 \land n + 18 = 4a + 7b$.

Base case: when $n=0$ we have $0 + 18 = 4 \cdot 2 + 7 \cdot 2$ which satisfies $P(0)$ by choosing $a=1$ and $b=2$.

Step case: Assume $P(n)$ holds.

Let $18 + n = 4a + 7b$ for some (non-negative) $a$ and $b$.

First we show that if $b = 0 \land a < 5$ then $4a + 7b < 4a + 0 < 16 < 18$. (Aside. that is, $b = 0 \land a < 5 \rightarrow 4a + 7b < 18$. Therefore using the contrapositive, since we have $y P(n)$ that $4a + 7b = 18 + n \geq 18$, we can deduce $\neg(b = 0 \land a < 5)$ which by De Morgan is equivalent to $b \geq 1 \lor a \geq 5$.

For showing $P(n + 1)$ there are two cases to consider: $b \geq 1$ and $a \geq 5$.

First consider the case that $b \geq 1$. Then Now $18 + n + 1 = 4a + 7b + 1$ by the induction hypothesis and $4a + 7b + 1 = 4a + 7b + 8 - 7 = 4(a + 2) + 7(b - 1)$ which satisfies $P(n + 1)$ by choosing $a$ is $a+2$ and $b$ is $b-1$. Note that both $a+2$ and $b-1$ will be $\geq 0$ as required.

Second consider that case that $a \geq 5$. Now $18 + n + 1 = 4a + 7b + 1$ by the induction hypothesis and $4a + 7b + 1 = 4a + 7b + 21 - 20 = 4(a - 5) + 7(b + 3)$ which satisfies $P(n+1)$ by choosing $a$ is $a-5$ and $b$ is $b+3$. Note that both $a - 5$ and $b + 3$ are $\geq 0$ as required.

Therefore by induction we have shown that $\forall n. \exists a, b. n + 18 = 4a + 7b$. QED.

For more practice questions, see Chapter 5 of *Mathematics for Computer Science*, and the resources listed on the CITS2211 website.

**Challenge Questions**

1. Consider a $2^n \times 2^n$ chessboard, with a single square removed from the top-right corner. Show that any such chessboard can be completely covered by L-shaped tiles as shown in the diagram.
(This is a more interesting induction proof as it is not just a simple statement about integers; however it is quite straightforward, and only needs weak induction.)

**Solution**  
**Statement:** Let $P(n)$ be the statement: A $2^n \times 2^n$ chessboard with any of its corners missing can be tiled with L-shaped tiles.

**Base case:**
When $n = 1$ the statement says that a $2 \times 2$ chessboard with one corner missing can be tiled with L-shaped tiles, and this is clearly true seeing that single L-shaped tile (as shown in the picture) can be used (perhaps rotated if another corner than the one on the picture is missing).

**Inductive Step:**
Suppose that $P(k)$ is true, and so a $2^k \times 2^k$ chessboard with any its corners missing can be tiled with L-shaped tiles. We need to use this fact to prove $P(k + 1)$. In other words, to show that a $2^{k+1} \times 2^{k+1}$ chessboard with any of its corners missing can be tiled with L-shaped tiles.

First we note that a $2^{k+1} \times 2^{k+1}$ board can be divided into four $2^k \times 2^k$ boards.

By the inductive hypothesis, each of these squares can be tiled with L-shapes missing any corner. So suppose that the top-left square has been tiled with its bottom right corner uncovered.

Now use the same tiling, but rotated, so that the bottom-left $2^k \times 2^k$ region is tiled except for its top-right corner.

Now use the same tiling, but rotated, so that the bottom-right $2^k \times 2^k$ region is tiled except for its top-left corner.

Finally, use the same tiling, but rotated, so that the top-right $2^k \times 2^k$ region is tiled except for its top-right corner. The result is as follows

This process has resulted in one L-shaped hole in the centre of the board that can be filled with a tile as shown below.

And there is one square missing from the $2^{k+1} \times 2^{k+1}$ board in its top right.

If we want to leave any other corner square free, then we simply use the same tiling, but rotated.

Hence by the Principle of Mathematical Induction, the statement $P(n)$ is true for all $n \geq 1$. QED
2. How far can you generalize the proof of Theorem 1.8.1 (below) that $\sqrt{p}$ is irrational? For example, how about $\sqrt{3}$? Can you generalize this with an inductive hypothesis?

Source: Problem 1.13. Mathematics for Computer Science

**Example**

We’ll prove by contradiction that $\sqrt{2}$ is irrational. Remember that a number is *rational* if it is equal to a ratio of integers—for example, $3.5 = 7/2$ and $0.1111 \cdots = 1/9$ are rational numbers.

**Theorem 1.8.1.** $\sqrt{2}$ is irrational.

**Proof.** We use proof by contradiction. Suppose the claim is false, and $\sqrt{2}$ is rational. Then we can write $\sqrt{2}$ as a fraction $n/d$ in lowest terms.

Squaring both sides gives $2 = n^2/d^2$ and so $2d^2 = n^2$. This implies that $n$ is a multiple of 2 (see Problems 1.11 and 1.12). Therefore $n^2$ must be a multiple of 4. But since $2d^2 = n^2$, we know $2d^2$ is a multiple of 4 and so $d^2$ is a multiple of 2. This implies that $d$ is a multiple of 2.

So, the numerator and denominator have 2 as a common factor, which contradicts the fact that $n/d$ is in lowest terms. Thus, $\sqrt{2}$ must be irrational.

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**SOLUTION:** Yes, the proof can be generalised for 3 as follows.

Proof by contradiction. Suppose $\sqrt{3}$ is rational. Then there exist $n$ and $d$ such that $\sqrt{3} = n/d$ in lowest terms. Thus $3d^2 = n^2$. Since $n^2$ is a multiple of 3, so $n$ must be a multiple of 3 and so $n^2$ is a multiple of 9.

Also since $3d^2 = n^2 = 9k$ we know $d^2$ is a multiple of 3. This implies that $d$ is a multiple of 3.

Now both $n$ and $d$ are multiples of 3 contradicting the assumption that $\sqrt{3}$ can be written as $n/d$ in lowest terms. Therefore $\sqrt{3}$ it must be irrational. QED.

In general, you need the property $n^2$ a multiple of $p$ implies $n$ is a multiple of $p$. This is true for 2 and 3, but obviously not for 4.