1. Express each of the statements in propositional logic using the propositions $A =$ “Alan likes discrete maths” and $B =$ “Bertrand likes discrete maths”.

   a. Alan likes discrete maths but Bertrand does not.
   b. Bertrand likes maths if Alan does too.
   c. Neither Alan nor Bertrand dislike maths

   **Solution:**
   
   a. $A \land \neg B$
   b. $A \rightarrow B$
   c. $\neg(\neg A \lor \neg B)$

      Note that this is a fairly literal translation; if asked or permitted to simplify, as well, then this simplifies to
      $A \land B$

2. Draw the truth tables for $P \lor Q$, $\neg P$ and $P \rightarrow Q$.

   **Solution:**

   $\begin{array}{c|c|c}
   P & Q & P \lor Q \\
   \hline
   T & T & T \\
   T & F & T \\
   F & T & T \\
   F & F & F \\
   \end{array}$

   $\begin{array}{c|c}
   P & \neg P \\
   \hline
   T & F \\
   F & T \\
   \end{array}$

   $\begin{array}{c|c|c}
   P & Q & P \rightarrow Q \\
   \hline
   T & T & T \\
   T & F & F \\
   F & T & T \\
   F & F & T \\
   \end{array}$

3. Define *exclusive or* using a truth table. Exclusive or is defined as either $P$ or $Q$ is
true but not both, eg. “your money or your life”.

**Solution:**

“$P \text{ xor } Q$” means either $P$ is true or $Q$ is true but not both.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \text{ xor } Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
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<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
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<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

4. Use a truth table to prove or disprove the following statement. Also give an example of an English language sentence corresponding to this statement.

$\neg (P \land (Q \lor R))$

is logically equivalent to

$(\neg P) \lor (\neg Q \lor \neg R)$. 
SOLUTION:

Hint: if the question specifically asks for a truth table, then make sure you use one. And if asked to prove or disprove a statement, be aware that it might be false, as in this case.

\[
\begin{array}{c|c|c|c|c|c|c}
P & Q & R & Q \lor R & (P \land (Q \lor R)) & \neg(P \land (Q \lor R)) \\
T & T & T & T & T & F \\
T & T & F & T & T & F \\
T & F & T & T & T & F \\
T & F & F & F & F & T \\
F & T & T & T & F & T \\
F & T & F & T & F & T \\
F & F & T & T & F & T \\
F & F & F & F & F & T \\
\end{array}
\]

It can be seen that the statements are NOT logically equivalent. They evaluate differently on the second and third row.

An example in English is as follows. Saying the two are equivalent would be like saying “Not having both a passport and either cash or a credit card is the same as not having either a passport, cash or a credit card.” (Which is not correct as a statement of logical equivalence.)

5. Use logical equivalence laws to prove or disprove the following statement.

\[\neg(P \lor (Q \land R)) \text{ is logically equivalent to } (\neg P) \land (\neg Q \lor \neg R)\]

SOLUTION:

Hint: if the question specifically asks you to use logical equivalence laws, then make sure you do so.

\[
LHS \equiv \neg(P \lor (Q \land R)) \\
\equiv (\neg P) \land (\neg(Q \land R)) \quad \text{by De Morgan} \\
\equiv (\neg P) \land (\neg Q \lor \neg R) \quad \text{by De Morgan}
\]

So they ARE logically equivalent.
Extension Questions

These questions are optional. They are provided to extend those who have completed the first part of the tutorial and would like something more interesting to get their teeth into.

Solution:
Please feel free to share your own answers for the extension questions. Email them to the unit co-ordinator and I will add them to the solution set.

1. The binary logical connectives \( \land \) (and), \( \lor \) (or), and \( \rightarrow \) (implies) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, \textit{nand}, which is simpler to represent in a circuit. Here is the truth table for \textit{nand}:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P nand Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

a. For each of the following expressions, find an equivalent expression using only \textit{nand} and \( \neg \) (not), as well as grouping parentheses to specify the order in which the operations apply. You may use \( A \), \( B \), and the operators any number of times.

i. \( A \land B \)

ii. \( A \lor B \)

iii. \( A \rightarrow B \)

b. It is actually possible to express each of the above using only \textit{nand}, without needing to use \( \neg \). Find an equivalent expression for \( \neg A \) using only \textit{nand} and grouping parentheses.

c. The constants true (\( T \)) and false (\( F \)) themselves may be expressed using only \textit{nand}. Construct an expression using an arbitrary statement \( A \) and \textit{nand} that evaluates to true regardless of whether \( A \) is true or false. Construct a second expression that always evaluates to false. Do not use the constants true and false themselves in your statements.
**Solution:**

a. i. \( A \land B \equiv \neg (A \text{ nand } B) \)
   
   ii. \( A \lor B \equiv (\neg A) \text{ nand } (\neg B) \)
   
   iii. \( A \rightarrow B \equiv A \text{ nand } \neg B \)

   (Check/Justify with truth tables).

b. \( \neg A \equiv (A \text{ nand } A) \)

c. As \( A \lor \neg A \) is always true, so is the following:

\[
(A \text{ nand } A) \text{ nand } A
\]

The negation of that is thus always false:

\[
(((A \text{ nand } A) \text{ nand } A) \text{ nand } ((A \text{ nand } A) \text{ nand } A))
\]

There are other possible correct answers.

Note that *nand* is not associative so you do need the parentheses.

2. Use the logical equivalence laws from lectures to show that

\[
(((P \land Q) \rightarrow R) \land (P \rightarrow Q)) \rightarrow (P \rightarrow R)
\]

is a tautology.
SOLUTION:
The aim is to simplify the expression to true. You can use the following (rather involved) strategy:

1. rewrite the implications to “not P or Q” but leave the outermost implication for now.
2. use de morgan to distribute not over and
3. take out the common ¬P from the two left hand side disjunctions
4. distribute the or over and again
5. now notice that ¬P ∨ R occurs on both the right and left hand sides,
6. for simplicity denote (¬P ∨ R) as A and denote (¬P ∨ Q) with B and
7. rewriting the last implication as a disjunction, we can see we have a tautology and finally true or anything gives true.

Thanks to Theo for his version similar to this. If anyone thinks of a tidier proof please let me know:

\[
\begin{align*}
(((P \land Q) \rightarrow R) \land (P \rightarrow Q)) & \rightarrow (P \rightarrow R) \\
\equiv & \quad (¬(P \land Q) \lor R) \land (¬P \lor Q) \rightarrow (¬P \lor R) \quad \text{(implication)} \\
\equiv & \quad (¬P \lor (¬Q \lor R) \land (¬P \lor Q)) \rightarrow (¬P \lor R) \quad \text{(de morgan)} \\
\equiv & \quad (¬P \lor (F \lor (R \land Q))) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor (F \lor (R \land Q))) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor (R \land Q)) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor ((¬Q \lor R) \land Q)) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor (A \land B)) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor (A \land B)) \rightarrow (¬P \lor R) \\
\equiv & \quad (¬P \lor (A \land B)) \lor A \\
\equiv & \quad ¬A \lor ¬B \lor A \\
\equiv & \quad T \lor B \\
\equiv & \quad T
\end{align*}
\]
3. A long time ago, in a country far away, the King sentenced a man to death. He told the man “You shall be shot at dawn, one day next week. However, the evening before you are shot, you will not know that you are going to be shot the next day.” While the King was a brutal dictator, he never told a lie. The condemned man proved that it was impossible for him to be executed. How did he do this and was he correct?

(Question is from Stueben, Michael and Diane Sandford. Twenty Years Before the Blackboard, Math. Assoc America, 1998 quoted by Prof. Albert R. Meyer and Prof. Ronitt Rubinfeld in MIT course 6.042J/18.062J)

4. See the “Lewis Caroll Logic Problems” under “Further Reading” on the cits2211 Resources web page. Convert the following statements into implication statements in propositional logic.

   a. No one studies logic, unless she is well educated.
   b. No hedgehogs can read
   c. Those who cannot read are not well educated.

Then, using transitive reasoning, derive a conclusion that uses all three statements. Give the deduced statement using propositional logic and also in words.

**Solution:**

Let \( L = \) studies logic, \( E = \) is educated, \( H = \) is a hedgehog, \( R = \) can read. Note that we have to assume these are about some implicit “person” (or “thing”, if hedgehogs aren’t people), since they aren’t quite propositions on their own.

   a. No one studies logic, unless she is well educated: \( \neg L \lor E, L \rightarrow E, \neg E \rightarrow \neg L \)

   b. No hedgehogs can read: \( R \rightarrow \neg H, H \rightarrow \neg R \)

   c. Those who cannot read are not well educated: \( \neg R \rightarrow \neg E, E \rightarrow R \)

From these, we can derive the chain of implications

\( L \rightarrow E \rightarrow R \rightarrow \neg H \)

and so can conclude

\( L \rightarrow \neg H \) that is, Those who study logic are not hedgehogs.

6. What if in addition to \( T \) and \( F \), we allowed identifiers to take a third value, \( U \), representing “unknown”? How would you construct the truth table for “not”? “and”? “or”? Could you give a truth table for “implies” which is closer to its usual meaning in English? What about “if and only if”? Try using your logic to see what logical equivalences still hold: does double negation hold? What about the law of the excluded middle? Try some others; do you see any difficulties with using your logic in practice?