1 Basic properties of integers

1.1. Prove that for any integer \( n \), if \( n > 1 \), then \( n^2 > n \). (E)

1.2. a. Give a direct proof that “If \( n \) is odd, then \( n^2 \) is odd.” (E)
   b. Now give a proof by induction of the same fact. (M)

1.3. Construct a contrapositive proof that for all real numbers \( x \), if \( x^2 - 2x \neq -1 \), then \( x \neq 1 \). (E)

1.4. Construct a proof that for all integers \( m \) and \( n \), if \( m \) is even and \( n \) is odd, then \( m + n \) is odd. (E)

1.5. Prove that for all integers \( m \) and \( n \), if \( m \) is odd and \( n \) is odd, then \( mn \) is odd. (E)

1.6. Give a proof by contradiction that if \( 3n + 2 \) is odd, then \( n \) is odd. (E)

1.7. Prove that the statement “All prime numbers are odd” is false. (E)

1.8. Prove that if \( n \) is a positive integer, then \( n \) is even if and only if \( 7n + 4 \) is even. (E)

1.9. Prove that \( m^2 = n^2 \) if and only if \( m = n \) or \( m = -n \). (E)

2 Basic properties of rational numbers

2.1. Prove that the sum of two rational numbers is rational. (M)

2.2. Prove that the product of two rational numbers is rational. (M)

2.3. Construct a proof by contradiction to show that the sum of an irrational number and a rational number is irrational. (M)
3 More complex proofs

3.1. Prove that \( n^3 - n \) is always divisible by 3. \((E)\)

3.2. Construct a proof by contradiction that for all real numbers \( x \), if \( x^2 - 2x \neq -1 \), then \( x \neq 1 \). \((E)\)

3.3. Prove that, for all integers \( n \), if \( n^3 + 5 \) is odd, then \( n \) is even, using

a. a proof by contradiction
b. a contrapositive proof

3.4. Prove that if \( x^3 > 8 \), then \( x > 2 \). \((E)\)

3.5. Prove that \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4} \). \((E)\)

3.6. Prove that every number greater than 7 can be expressed as the sum of a non-negative integer multiple of 3, and a non-negative integer multiple of 5. \((M)\)

3.7. Prove by induction that the number of subsets of an \( n \)-element set is \( 2^n \). \((M)\)

3.8. Prove that every square number is either (a) a multiple of 5, (b) a multiple of 5 plus 1, or (c) a multiple of 5 minus 1. \((VH)\)

3.9. Prove that for any integer \( n \), \( n^{10} - 1 \) is a multiple of 11. \((M)\)

4 Proofs by contradiction

Use proof by contradiction to prove the following. For each of these, consider also how you might prove them using either a direct or a contrapositive proof.

4.1. Prove that for all \( n \in \mathbb{Z} \), if \( n \) is odd, so is \( n^2 \). \((E)\)

4.2. Prove that for all \( n \in \mathbb{Z} \), if \( n^2 \) is odd, so is \( n \). \((E)\)

4.3. Prove that for all \( n \in \mathbb{Z} \), \( n^2 \) is odd if and only if \( n \) is odd. \((E)\)

4.4. Prove that for all \( a, b \in \mathbb{Z} \), \( a^2 - 4b - 2 \neq 0 \). \((M)\)

4.5. Prove that for all \( a, b, c \in \mathbb{Z} \), if \( a^2 + b^2 = c^2 \), then \( a \) or \( b \) is even. \((M)\)

5 Proofs by induction

5.1. Prove that \( \sum_{i=1}^{n} i(i + 1) = \frac{n(n + 1)(n + 2)}{3} \). \((E)\)

5.2. Prove that, for any positive integer \( n \), \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \). \((E)\)

5.3. Prove that for all integers \( n \), \( 2^n > n \). \((E)\)

5.4. Prove that for all integers greater than or equal to 4, \( 2^n \geq n^2 \). \((M)\)
5.5. Prove by induction that it is possible to pay, without requiring change, any whole number of roubles greater than 7 with banknotes of value 3 roubles and 5 roubles. \( (M) \)

5.6. Prove that for integers \( n \geq 0 \), the number \( 5^{2n} - 3^n \) is a multiple of 11. \( (M) \)

5.7. Prove that for any integer \( n \geq 1 \), the integer \( 2^{4n-1} \) ends with an 8 (when written in the base 10 number system). \( (M) \)

5.8. Prove that the sum of the cubes of three consecutive positive integers is always a multiple of 9. \( (M) \)

5.9. Prove that, if \( x \geq 2 \) is a real number and \( n \geq 1 \) is an integer, then \( x^n \geq nx \).

5.10. Prove that, if \( n \geq 3 \) is an integer, then \( 5^n > 4^n + 3^n + 2^n \).

5.11. A formal language to be used for simple algebra uses the symbols

\[ \text{x y z ( ) + } \]

The formulas of the language are strings of symbols formed according to the following rules:

i. x, y and z are formulas.

ii. If A and B are formulas, so is \((A)(B)\).

iii. If A and B are formulas, so is \((A) + (B)\).

For example, \(((x)(z)) + (z)\) is a formula, but \((x) + z\) is not.

Prove that any formula in this language has the same number of left parentheses, “(”, as it does right parentheses “)”. \( (M) \)

5.12. Prove by induction that \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \) \( (E) \)

### 6 Sources

Sources include: