CITS2211: Test Two 2019 SAMPLE SOLUTIONS

This test has 5 questions with a total value of 30 marks. Time allowed: 30 minutes.

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**SOLUTION:** Model answers, marking scheme and general feedback.
1. Write predicate logic formulas for the following statements. Use the predicates $F(x)$, “$x$ is fast”, and $G(x)$, “$x$ is furious”. If you need to define any constants, state what they are.

(i) Everything is fast.
(ii) No fast thing is furious.

**Solution:**

(i) Everything is fast : $\forall x. (F(x))$

(ii) No fast thing is furious:

$$\neg \exists x. (F(x) \land G(x))$$

(literally: “No things exist such that they are both fast and furious”.)

Or alternatively:

$$\forall x. (F(x) \rightarrow \neg G(x))$$

(literally: If a thing is fast, it cannot be furious.)

The two are equivalent, as can be seen by applying de Morgan’s laws, and applying the implication equivalence.

Marking rubric:

- 2 marks for part (i)
- 4 marks for part (ii)
- $\frac{1}{2}$ to 1 mark off for minor errors
- 2 marks off for major errors

**Marker’s comments:**

Quite a few people seemed to treat the predicates $F$ and $G$ like functions instead of predicates – saying things like $G(F(x))$.

You can’t do this: review the definition of a predicate, and of predicate logic formulas in the lecture on predicate logic, and in section 3.6 of the MCS textbook.

If we have a predicate like $F$, then we can supply it with arguments which are constants or variables. So, $F(a)$ or $F(x)$ would be okay. But we can’t supply it with another predicate, so $F(G)$ is wrong; neither can we supply it with (say) a whole predicate logic formula, so $F(G(x))$ is wrong also, as would be something like $F(G(x) \land H(y)))$.  

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2. For each of the Venn diagrams below, write down a set theory expression which equals the shaded area of the diagram.

(i)

\[ ((A \cap B) \cup (A \cap C) \cup (B \cap C)) - (A \cap B \cap C) \]

**Solution:**
One possible expression is: \((A \cap B) \cup (A \cap C) \cup (B \cap C)) - (A \cap B \cap C)\)

**Marker's comments:**
The most common mistake was forgetting to subtract \(A \cap B \cap C\) from this expression. Another was making use of the complement symbol. Refer to the lecture slides on set theory, and to section 4.1.2 of the MCS textbook. We can only use the complement symbol *if we’re told that there’s a universal set*. If we aren’t told there’s a universal set, then we can’t use it.

(ii)

\[ A \cup (B \cap C) \]

**Solution:**
One possible expression is: \(A \cup (B \cap C)\)

**Marking rubric:**
- 4 marks for part (i)
• 2 marks for part (ii)
• $\frac{1}{2}$ to 1 mark off for minor errors
• 2 marks off for major errors
3. For each of the following, give an example of a relation that meets the specified criteria, explaining briefly why it meets those criteria.

(i) A binary relation on the natural numbers that is reflexive.

**Solution:**
There are many examples, for instance:

- The relation “less than or equal to” – since every number is less than or equal to itself.
- The identity relation, which relates each number to itself and nothing else. (i.e., $R = \{(x, x) \mid x \in \mathbb{N}\}$)

(ii) A binary relation on the integers which is symmetric but not reflexive.

**Solution:**
An example might be the relation $R$ on $\mathbb{Z}$ where $R = \{(x, y) \mid x \neq y\}$ (i.e., the “not equals” relation).

It’s not reflexive, because no integer is unequal to itself (on the contrary, every integer is equal to itself).

But it is symmetric, since if for two integers $x$ and $y$ it’s true that $x \neq y$, then it’s also true that $y \neq x$.

(iii) An antisymmetric relation on the real numbers.

**Solution:**
An example would be the “less than or equal to” relation.

For any two real numbers $x$ and $y$, if $x \leq y$ is true and $y \leq x$ is also true, then $x$ must equal $y$; which is exactly the condition required for antisymmetry.

(Alternative formulation of this: if $x \neq y$, then only one of $x \leq y$ or $y \leq x$ can be true, not both.)

**Solution:**
Marking rubric:

- 2 marks for each part
- $\frac{1}{2}$ mark off for minor errors
- 1-2 marks off for major errors
- Failing to give an explanation is a major error

**Marker’s comments for (i)–(iii):**
Common mistakes included incorrectly applying the definition for “reflexivity”.

Refer to the lecture slides for the definition of reflexivity:
A relation $R$ on the set $A$ is reflexive if, for all $a \in A$, $(a, a) \in R$. Thus the relation $\{(1, 1), (2, 2), (3, 3)\}$ on the natural numbers is not reflexive. There are elements of $\mathbb{N}$ which
don’t appear paired with themselves in the relation; thus it is not reflexive. (For instance, 5 is in \( \mathbb{N} \), but \((5, 5)\) is not in \(R\).)

Another was giving answers in terms of there being “loops” or not. It was stressed in the lecture on set theory: **stating the mnemonics is not acceptable as an answer**, and does not count as showing something is e.g. reflexive. The mnemonics are **only there as an aide to understanding**. The **definitions** are the only things you can validly refer to when giving an answer to a question about the properties of a relation.
4. Consider the two sets $R$ and $S$, where

\[
R = \{ \emptyset, \{ \emptyset, 2, \{ 5, 7 \} \} \}
\]

and

\[
S = \{ \{ \emptyset, 2, \{ 5, 7 \} \}, \emptyset, \{ 1, 2, 5, 7 \} \}
\]

Answer the following questions about them, giving a brief explanation.

(i) Which of the sets contain the empty set as an element?
(ii) Which of the sets have the empty set as a subset?
(iii) Which set has the greater number of numbers?
(iv) Is $S$ a subset of itself?

**Solution:**

(i) They both do – it is explicitly listed.
(ii) They both do, since every set has the empty set as a subset.
(iii) The second set ($S$) has the greater number of members – it has 3, whereas $R$ has only 2.
(iv) $S$ is a subset of itself, because every set is a subset of itself.

**Marking rubric:**

- 1.5 marks for each part
- 0.5 marks penalty for a minor error
- 1-1.5 marks penalty for a major error
- Failing to give an explanation for a part results in 0 marks for that part.

**Marker’s comments:**

Many students seem confused about the difference between subsets of a set and elements of a set, and about the empty set.

If you didn’t do well on this question, you should:

- review slides 22–24 of the lecture on set theory, and the relevant exercises
- ensure you’ve worked through the relevant parts of Chapter 4 of the MCS textbook
5. Consider the following proof:

**To be proved:** If \( n \) is a square number, then either it is divisible by 4, or there is some number \( x \) such that \( n = 4x + 1 \).

Let \( n \) be some square number. Then there is some number \( k \) such that \( n = k^2 \).

Now, when we divide \( k \) by 4, the remainder is either 0, or 1, or 2, or 3. Suppose it is 0. Then there is some \( q \) such that \( k = 4q \), and \( n = k^2 = (4q)^2 = 16q^2 \), and \( n \) is a multiple of 4. Suppose on the other other hand the remainder is 1. Then there is some \( q \) such that \( k = 4q + 1 \). If that’s so, then \( n = k^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q + 1) + 1 \). So \( n \) gives a remainder of 1 when divided by 4. Alternatively, suppose the remainder is 2. Then there’s some \( q \) such that \( k = 4q + 2 \), and \( n = k^2 = (4q + 2)^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1) \), which is divisible by 4. Finally, suppose instead the remainder is 3. Then there is some \( q \) such that \( k = 4q + 3 \), and \( n = k^2 = (4q + 3)^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1 \), which gives a remainder of 1 when divided by 4.

So regardless of the remainder \( k \) gives when divided by 4, \( n \) will either be divisible by 4, or will leave a remainder of 1 when divided by 4. Therefore this must be true of all squares. ■

Answer the following questions about the proof:

(i) Write a predicate logic expression for the assertion being proved. (i.e., “If \( n \) is a square number, then either it is divisible by 4, or there is some number \( x \) such that \( n = 4x + 1 \).”)

(ii) Of the proof techniques we have seen (direct, contrapositive, by cases, by contradiction, by induction), which are used in this proof? Explain your answer.

(iii) Does this proof make use of universal elimination? If so, where?

**Solution:**

(i) \( \forall n. (\exists k. (n = k^2) \rightarrow \exists j. ((n = 4j) \lor (n = 4j + 1))) \)

Explanation:
Let’s define the predicate \( S(n) \) as meaning “\( n \) is a square number”. Then \( S(n) \) is written as: \( \exists k. (n = k^2) \)
Let’s define the predicate \( P(n) \) as meaning “\( n \) is either divisible by 4, or leaves a remainder of 1 when devided by 4”. Then \( P(n) \) is written as: \( \exists j. ((n = 4j) \lor (n = 4j + 1)) \).
The statement as a whole is then \( \forall n. S(n) \rightarrow P(n) \), which expands to exactly the formulation given above.

(ii) It makes use of proof by cases – it considers the case where \( k \) divided by 4 leaves a remainder of 0, 1, 2 or 3, and shows that in all cases, \( n \) must either be divisible by 4 or leave a remainder of 1.
   It also make use of use direct proof, since it uses the assumption that \( n = k^2 \), and then proceeds to prove that the divisibility property follows from that.
   The proof does not make use of contrapositive proof, proof by contradiction or proof by induction.

(iii) Universal elimination is where we already know some “for all” statement, and we already have some constant thing in the domain of discourse, and we reason as follows:
   - Suppose we know \( \forall x. (P(x)) \). And suppose we have some constant \( a \). Then since \( P \) applies to all things, we can deduce \( P(a) \).
Universal *introduction*, on the other hand, is where we show that, given some arbitrary thing in the domain of discourse, we can prove a predicate $P$ is true of it; it therefore follows that $P$ is true of all things, i.e. $\forall x. (P(x))$. So the reasoning goes:

- Suppose we have shown $P(a)$ is true for some arbitrary $a$. Then $\forall x. (P(x))$.

For this question, marks are awarded if a justification is given in terms of the inference rules.

The best example of a use of the “universal elimination” rule is actually the statement “Now, when we divide $k$ by 4, the remainder is either 0, or 1, or 2, or 3.” This expresses a general truth about numbers and their remainders when divided by 4 (we might call this property of numbers $P(n)$). When we say that the property is true of $k$, we’re performing universal elimination. But this is quite tricky to spot, so as long as a reasonable answer is given in terms of the inference rules, that’s acceptable.

Marking rubric:

- 2 marks for each part.
- $\frac{1}{2}$ mark off for minor errors
- 1-2 marks off for major errors
- For part (ii), not identifying “proof by cases” is a major error, as is incorrectly stating that proof by contrapositive, contradiction or induction is used. The “case” structure here should be very clear. Not mentioning direct proof is only a minor error, however. It’s true that direct proof is used, but the major structural element of the proof is its use of proof by cases.