1. Use a truth table to prove or disprove the following statement.

\[\neg((P \lor Q) \land R)\] is logically equivalent to \((\neg P) \land (\neg Q \land \neg R)\)

**SOLUTION:**

<table>
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<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>(P \lor Q)</th>
<th>((P \lor Q) \land R)</th>
<th>LHS</th>
<th>\neg P</th>
<th>\neg Q</th>
<th>\neg R</th>
<th>\neg Q \land \neg R</th>
<th>RHS</th>
<th>LHS \leftrightarrow RHS</th>
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</table>

There are some \(F\) values in the last column so the formulas are not logically equivalent.

You can stop filling in the table after the second line and still get full marks.

**FROM THE MARKER:**

I did not in general deduct marks for minor errors in the truth table. In particular if the student said: "not equivalent due to row 2", I would not check the correctness of other rows as they were not relevant.

On some tests I would put "Why/Because/See e.g. row 2", This is because I thought it would be ideal if the student in some sense highlighted the row or at least wrote e.g. "LHS+RHS differ". I did not deduct marks for this if the truth table was correct.

Many students spent too much time on this question. Most could have quit after the second row and still disproven equivalence.
2. Which of the following are tautologies? Give brief reasons for each answer.

(a) \( P \rightarrow P \)
(b) \( (P \lor Q) \rightarrow P \)
(c) \( (P \lor \neg P) \rightarrow P \)
(d) \( (Q \land ((P \land Q) \lor (\neg P \land \neg Q))) \rightarrow P \)
(e) \( (P \land \neg P) \rightarrow P \)

**Solution:**

(a) \( P \rightarrow P \) **IS** a tautology. True implies True, and False implies False.

(b) \( (P \lor Q) \rightarrow P \) **is NOT** a tautology. Eg, if \( Q \) is true and \( P \) is false.

(c) \( (P \lor \neg P) \rightarrow P \) **is NOT** a tautology. Eg, if \( P \) is false.

(d) \( (Q \land ((P \land Q) \lor (\neg P \land \neg Q))) \rightarrow P \) **IS** a tautology.

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<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg P )</th>
<th>( \neg Q )</th>
<th>( P \land Q )</th>
<th>( \neg P \land \neg Q )</th>
<th>( (P \land Q) \lor (\neg P \land \neg Q) )</th>
<th>( Q \land \ldots )</th>
<th>( \ldots \rightarrow P )</th>
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(e) \( (P \land \neg P) \rightarrow P \) **IS** a tautology. False implies anything.

**FROM THE MARKER:**

One mark per sub question. Up to one mark penalty for failing give sufficient reasons. A total of at least two marks if gave a clear correct answer to one subquestion without including anything wrong.

in later tests and exams, randomly guessing without reason may not be expected to gain many marks.
Consider the following four propositions forming an argument.

If Tiexia is sick then he is absent.
If Tiexia is absent then there will be no test with his name on it.
There is a test with Tiexia’s name on it.
Therefore, Tiexia is not sick.

Using variables for simple propositions and some technique that we have studied show that the conclusion must be true if the first three propositions are.

**Solution:**

Use the following propositions:

$S = \text{Tiexia is sick.}$

$A = \text{he is absent.}$

$N = \text{there is a test with his name on it.}$

So we are told that the following are true: $S \rightarrow A$, $A \rightarrow \neg N$, and $N$.

The conclusion is $\neg S$.

Here are all the possible situations in a truth table:

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<th>6</th>
<th>7</th>
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<tr>
<td>$S$</td>
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<td>$S \rightarrow A$</td>
<td>$\neg N$</td>
<td>$A \rightarrow \neg N$</td>
<td>$\neg S$</td>
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The only time that columns 4 ($S \rightarrow A$), 6 ($A \rightarrow \neg N$) and 3 ($N$) are true is is given by the second last row. And in that row $\neg S$ (column 7) is also true.

This shows what is required.

There are other ways to show this.

**FROM THE MARKER:**

5 Marks: Used a logic technique as described the lectures to prove. 4 Marks: Used a mathematical technique to prove. 3 Marks: No proof, but on right track (e.g. specification with no unnecessary use of predicate logic) 2 Marks: Correct Specification, did not seem to be on right track (e.g. use of predicate logic) 0/1 Marks: More serious problem, e.g. attempt to quantify over Tiexa.
4. Recall that conjunction distributes over disjunction, i.e.

\[(A \lor B) \land C\] is logically equivalent to \[(A \land C) \lor (B \land C)\]

Consider the following formula of propositional logic.

\[(P_{1,1} \lor P_{1,2} \lor P_{1,3}) \land (P_{2,1} \lor P_{2,2}) \land (P_{3,1} \lor P_{3,2})\]

Here each \(P_{i,j}\) is just a simple proposition which is true or false.
This formula has 19 symbols as you count the \(P_{i,j}\) as just one symbol (and counting parentheses).
This is called a conjunction (\(\land\)) of disjunctions (\(\lor\)).
Find a logically equivalent formula which is a disjunction of conjunctions. You don’t need to write it all out but tell us how many symbols it has.

**SOLUTION:**

\[(P_{1,1} \land P_{2,1} \land P_{3,1}) \lor ... \lor (P_{1,3} \land P_{2,2} \land P_{3,2})\]

of length 95 because there are twelve disjuncts each one of length 7, plus eleven disjunction signs.
There are 12 = 3 \cdot 2 \cdot 2 disjuncts because we choose one \(P_{1,x}\) from 3, one \(P_{2,x}\) from 2 and one \(P_{3,x}\) from 2, to be a conjunct to make each disjunct.
The formulas are equivalent by repeated use of the distribution rule.

FROM THE MARKER:
Student found question 4 difficult and usually omitted or just made some nonsense guess.
5 Marks: 95 4 Marks: Uses negations, otherwise correct answer (uses de Morgan’s instead of distributivity) 3-4 Marks: not 95, on the right track 2 Marks: Mentions distributivity 1 Mark: Mentions de Morgan’s law

143 was a popular answer, sometimes without working. I did not give marks for 143 without working.
5. Express the following colloquial English statements using predicate logic. (First identify the predicates.)

(a) All data scientists have application domain knowledge.
(b) No hackers have application domain knowledge.
(c) Some software developers are data scientists.
(d) No data scientists are hackers.
(e) Although all data scientists are software developers, some are hackers if they do not have application domain knowledge.

**Solution:**

Use the following propositions:

\[ D(x) = x \text{ is a data scientist.} \quad K(x) = x \text{ has application domain knowledge.} \quad H(x) = x \text{ is a hacker.} \quad S(x) = x \text{ is a software developer.} \]

(a) \( \forall x. (D(x) \to K(x)) \)
(b) \( \neg \exists x. (H(x) \land K(x)) \)
(c) \( \exists x. (S(x) \land D(x)) \)
(d) \( \neg \exists x. (D(x) \land H(x)) \)
(e) \( (\forall x. (D(x) \to S(x))) \land (\exists x. (D(x) \land (\neg K(x) \to H(x)))) \)

**FROM THE MARKER:**

1) Does not use predicates, otherwise plausible logic 2) Uses Predicates only or uses Predicates and quantifiers, but with major problems 3) Uses quantifiers, logic broadly correct, some quantifiers may be missing 4) Uses quantifiers where appropriate 5) Correct/Plausibly Correct
6. This question is about the set of all integers. Suppose that we colour in some integers red and some green. Write \( R(n) \) or \( G(n) \) in those cases.

Translate the following properties into normal English.

(a) \( G(0) \)

(b) \( \forall x. \neg(G(x) \land R(x)) \)

(c) \( \exists x. \exists y. \exists z. (x < y \land y < z \land G(x) \land R(y) \land G(z)) \)

(d) \( \forall x. (G(x) \rightarrow (\neg\exists y. (x < y \land R(y)) \lor (\forall y. (x < y \rightarrow R(y)))) \)

**Solution:**

(a) Zero is green.
(b) No number is both red and green.
(c) There is a red number somewhere between two green ones.
(d) If any number is green then either there is no bigger red number or all bigger numbers are red.

**FROM THE MARKER:**

1 Mark for demonstrating understanding of predicate logic +1 per correctly answered subquestion