We are asked to prove “If \( n \) is even, then \( n^2 \) is even”. (The domain is not stated, but we will assume it is the integers, \( \mathbb{Z} \).)

More formally, what we’re asked to prove is:

\[ \forall n. \ ( \text{Even}(n) \rightarrow \text{Even}(n^2) ) \]

where \( \text{Even}(x) = \text{“}x \text{ is even}\text{”} \).

We proceed as follows.

1. Suppose we have some arbitrary integer, call it \( a \). We would like to show that \( \text{Even}(a) \rightarrow \text{Even}(a^2) \).

   (This is the first step in “for all” introduction – assume an arbitrary thing in the domain.)

1.1. To show that \( \text{Even}(a) \rightarrow \text{Even}(a^2) \), we will start by assuming that \( \text{Even}(a) \); then if we can derive \( \text{Even}(a^2) \) from that assumption, that’s enough to prove \( \text{Even}(a) \rightarrow \text{Even}(a^2) \).

1.1.1. So, assume \( \text{Even}(a) \).

   (This is the first step in a direct proof. If we want to prove \( P \rightarrow Q \), we assume \( P \) and show that we can derive \( Q \).)

1.1.2. If \( \text{Even}(a) \) is true (i.e., \( a \) is even), that means it must be two times some number.
(This is the definition of “even”: \( \text{Even}(n) = \exists k. (2k = n) \). This is something mathematicians just take for granted, barely bothering to spell out, but we do so here.)

Let’s call that number \( b \).

(Existential elimination. We’ve used a new name, \( b \), for the number we know must exist, which is half of \( a \).)

1.1.3. We know that \( a = 2b \). Therefore, \( a^2 = (2b)^2 \).

(An equivalence from mathematics. If we know two things are equal, we can square both sides, and the results will also be equal.)

And therefore \( a^2 = 4b^2 \).

(Another equivalence from mathematics. \( (xy)^t = x^t y^t \).)

1.1.4. Let \( c = 2b^2 \).

(This is defining something. It is purely for convenience. We are defining \( c \) as \( 2b^2 \), so we can make our proof shorter and clearer.)

Then

\[
a^2 = 4b^2 = 2c
\]

1.1.5. That means that \( 2c \) is even.

(From the definition of “even”. Since there exists a number \( c \) which is half of \( 2c \), it follows that \( \text{Even}(2c) \).)

1.1.6. \( a^2 = 2c \), and we just showed that \( 2c \) is even, so that means \( a^2 \) is even.

(From the way mathematical equality works. If \( x = y \), and something is true of \( x \), it must be true of \( y \).)

1.2. By assuming \( \text{Even}(a) \), we were able to infer \( \text{Even}(a^2) \). Therefore \( \text{Even}(a) \to \text{Even}(a^2) \).

(Steps 1.1.1 through 1.1.6 are where we inferred \( \text{Even}(a^2) \) from \( \text{Even}(a) \). So now we consider \( \text{Even}(a) \to \text{Even}(a^2) \) proved.)
1.3. Therefore, we’ve proved that, for an arbitrary integer $a$, $\text{Even}(a) \rightarrow \text{Even}(a^2)$.

(We are now finished with the steps needed for “for all” introduction.)

2. Given an arbitrary integer $a$, we were able to show $\text{Even}(a) \rightarrow \text{Even}(a^2)$.

Therefore, $\forall n. (\text{Even}(n) \rightarrow \text{Even}(n^2))$;

which is what we were asked to prove.

(Our proof is done. We end a proof with either the abbreviation “QED” – roughly, “which is what we set out to prove”, in Latin – or with a box, usually filled in, called a “Halmos block” or “tombstone”, and introduced by Paul Halmos – https://en.wikipedia.org/wiki/Tombstone_(typography).)

Other comments: In many mathematical proofs, the author might say something like “Let $n$ be an arbitrary integer”, assuming that the reader can work out from context whether $n$ is a variable or has become a constant – a named thing, which we can use like any other constant.

Here, we stick to the convention from week 2 that constants come from the start of the alphabet ($a$, $b$, $c$ etc.) and variables are $m$ through $z$. We loosened the requirement that predicates are named with capital letters, and used $\text{Even}(x)$ to refer to the property of being even. We could’ve abbreviated it as, say, $E$ or $P$, but then the proofs become harder to read. In general, the more formal an argument is, usually the less readable (by humans).